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Pluricomplex Green Functions with Logarithmic Poles at Infinity

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Contents

Abstract vii

Árip (in Icelandic) viii

Acknowledgements ix

1 Introduction 1

- 1.1 Historical background . . . 2
  - 1.1.1 Polynomial approximation . . . 2
  - 1.1.2 The Dirichlet problem . . . 3
  - 1.1.3 Walsh approximation theorem . . . 4
  - 1.1.4 Subharmonic functions and the Perron method . . . 5

2 The pluricomplex Green function 7

- 2.1 Plurisubharmonicity . . . 7
  - 2.1.1 Semicontinuity and subharmonicity . . . 7
  - 2.1.2 Plurisubharmonicity . . . 10
  - 2.1.3 Polynomials, homogeneity and balanced sets . . . 13
- 2.2 The Lelong class . . . 16
- 2.3 Pluricomplex Green function with logarithmic pole at infinity . . . 21
- 2.4 Continuity of the pluricomplex Green functions . . . 24

3 Siciak's approximation theorem 27

- 3.1 Equivalence of Siciak's and Zahariuta's definitions . . . 28
- 3.2 Siciak's approximation theorem . . . 29

4 Disc formulas for the weighted Green function 35

- 4.1 Analytic discs and disc functionals . . . 36

4.1.1 Notation and basic properties 36
4.1.2 Construction of the disc functional 37
4.1.3 Good sets of analytic discs 39
4.1.4 The good set $\mathcal{B}_X$ of analytic discs 40
4.2 A disc formula for the weighted Green function 41
4.2.1 Conclusion 47
4.3 Applications of the disc formula 48

A Index of symbols 51

Abstract

In this thesis we study the *pluricomplex Green function with logarithmic pole at infinity*, $V_{X,q}$, for a subset X of $\mathbb{C}^n$ and a weight function q on X. We study the continuity properties of $V_{X,q}$ with respect to X and q. If K is compact and q continuous we show that $V_{K,q}$ coincides with $\log \Phi_{K,q}$, where $\Phi_{K,q}$ is *Siciak's extremal function.*

We prove Siciak's approximation theorem for holomorphic functions, which gives a necessary and sufficient condition for a holomorphic function to have a holomorphic extension to a sublevel set of the Green function and the largest sublevel set where an extension exists.

A proof of a disc formula for the Green function is presented when X is an open domain and q upper semicontinuous. This allow us to evaluate the Green function using analytic discs in the complex projective space $\mathbb{P}^n$.

Ágrip

Í þessari ritgerð skoðum við *Green-fallið með lograskaut í óendanlegu*, $V_{X,q}$, fyrir hlutmengi X í $\mathbb{C}^n$ og vigt q, sem er fall á X. Við skoðum samfelldniskilyrði fyrir $V_{X,q}$ bæði með tilliti til X og q. Ef K er þjappað og q samfellt þá sýnum við að $V_{K,q}$ er jafnt $\log \Phi_{K,q}$, þar sem $\Phi_{K,q}$ er *útgildisfall Siciaks.*

Við sönnum nálgunarsetningu Siciaks fyrir fáguð föll, en hún gefur nauðsynleg og nægjanleg skilyrði fyrir fáguð föll svo þau hafi fágaða framlengingu yfir á undirgildismengi Green-fallsins og stærsta undirgildismengið þar sem slík framlenging er til.

Loks sönnum við skífuformúlu fyrir Green-fallið þegar X er opið svæði og q hálfsamfellt að ofan. Þetta gefur okkur aðferð til þess að reikna út gildi Green-fallsins með fáguðum skífum í tvinnvarprúminu $\mathbb{P}^n$.

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1

Introduction

In this thesis we start by defining the *pluricomplex Green function with logarithmic pole at infinity*, $V_{X,q}$, for a subset X of $\mathbb{C}^n$ and a function q on X. The function $V_{X,q}$ is also called the *Siciak-Zahariuta extremal function* and the *global extremal function* for X and q. First we have to state some necessary properties of plurisubharmonic functions and introduce the Lelong class of plurisubharmonic functions. This enables us to study the continuity properties of $V_{X,q}$ with respect to the set X and the function q, which is the main subject of Chapter 2.

In Chapter 3 we show that $V_{K,q}$ coincides with $\log \Phi_{K,q}$, when K is a compact set and q is continuous, where $\Phi_{K,q}$ is *Siciak's extremal function* introduced in [21]. Then we prove Siciak's approximation theorem for holomorphic functions, which is the main catalyst for the studies of the pluricomplex Green function. The theorem gives a necessary and sufficient condition for a holomorphic function to have a holomorphic extension to a sublevel set of the Green function and it gives the largest sublevel set where an extension exists.

Finally, in Chapter 4, we present a proof of a disc formula for the Green function when X is an open domain and q is an upper semicontinuous function on X. This allows us to evaluate the Green function using analytic discs in the complex projective space $\mathbb{P}^n$. The disc formula is a generalization of a formula proved by Lárusson and Sigurdsson [13], which gives a disc formula for the pluricomplex Green function when $q = 0$. This work continues the theory of disc functionals initiated by Poletsky in [18]. Disc functionals give a new way to look at extremal functions such as the pluricomplex Green function: instead of taking the supremum over a class of plurisubharmonic functions with some suitable properties, we look at the infimum of a disc functional over a class of holomorphic discs.

1.1 Historical background

1.1.1 Polynomial approximation

When studying polynomial approximation of functions defined on a subset of $\mathbb{R}^n$, the *Weierstraß approximation theorem* gives a very strong result. It states that if f is a continuous function on a compact subset K of $\mathbb{R}^n$, then for all $\varepsilon > 0$ there exists a polynomial P in real variables which satisfies $|f(x) - P(x)| < \varepsilon$ for all $x \in K$.

When we look at functions defined on subsets of $\mathbb{C}^n$ we get less satisfactory results. Then Weierstraß's approximation theorem fails, since it is not possible to approximate all continuous functions defined on a compact subset of $\mathbb{C}^n$ with polynomials in complex variables. However, a classical result from every elementary course on complex analysis states that a holomorphic function on a disc in $\mathbb{C}$ can be approximated uniformly on compact sets of the disc by partial sums of its power series. The natural question arises if this holds for more general sets in $\mathbb{C}$. The *Runge theorem* gives an answers to this question, in its simplest form it says the following:

If K is a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected, then every function holomorphic in an open neighbourhood of K can be approximated uniformly on K by polynomials.

The Runge theorem comes in many different versions, giving a wide variety of results regarding both polynomial approximation and approximation with rational functions, see Remmert [19] Chapters 12-13. Note that the conditions in Runge's theorem are only topological.

There exists also a Runge theorem for several variables. Then conditions on the set K are not purely topological as in one dimension but depend also on pseudoconvexity, see Krantz [10] Corollary 5.4.3.

It is a natural question whether the conditions in Runge's theorem can be relaxed, is it sufficient that f is holomorphic inside of K and continuous on K? Almost seventy years after Runge's theorem appeared, Mergelyan showed that this was the case, see Greene and Krantz [2] Theorem 12.2.1. His theorem states the following:

If K is a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected, then every function continuous on K and holomorphic on the interior of K can be approximated uniformly on K by polynomials.

We will show that polynomial approximations are tied to holomorphic functions in another way than described in Runge's theorem. That is, how this approximation can help us giving some answers to the important question: If f is a holomorphic function in a neighbourhood of a compact set $K \subset X$, what is the largest open set to which f can be extended holomorphically? It turns out that this is closely related to the polynomial approximations of f on K.

1.1.2 The Dirichlet problem

Another problem from complex analysis of one complex variable is the classical Dirichlet problem. If Ω is a bounded open subset of $\mathbb{C}$ and $f \in \mathcal{C}(\partial\Omega)$, then we seek a function u harmonic on Ω such that $u|_{\partial\Omega} = f$. This problem can also be defined for the compliment of Ω, assuming $f = 0$, it is of the following form:
For a compact set K in $\mathbb{C}$, find a function $g_K : \mathbb{C} \to \mathbb{R}_+$ such that

$$\begin{aligned}
&\Delta g_K = 0 \text{ on } \mathbb{C} \setminus K,\\
&g_K|_{\partial K} = 0,\\
&g_K(z) \sim \log|z| \text{ when } |z| \to \infty.
\end{aligned}$$

This can be formulated using the classical Dirichlet problem for bounded sets, because finding g_K is equivalent to finding a function w_K such that $\Delta w_K = 0$ on $\hat{\mathbb{C}} \setminus K$ and $w_K(z) = -\log|z|$ for $z \in \partial K$. Then $g_K(z) = \log|z| + w_K(z)$. Here we let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, or the complex projective space of dimension one, $\mathbb{P}^1$. The space $\mathbb{P}^n$ is defined as the set of all complex lines in $\mathbb{C}^{n+1}$. That is, we define the equivalence relation $\sim$ on $\mathbb{C}^{n+1} \setminus \{0\}$ such that $z \sim w$ if and only if $z = \lambda w$ for some $\lambda \in \mathbb{C}$, then $\mathbb{P}^n$ is defined as the set of all equivalence classes.

The simplest example of the Dirichlet problem is when K is the closed unit disc $\overline{\mathbb{D}}$, then obviously $g_{\overline{\mathbb{D}}}(z) = \log|z|$.

When working in $\mathbb{C}$ we have the powerfull *Riemann mapping theorem* which states that if U is a simply connected open domain of $\hat{\mathbb{C}}$ such that the compliment $\hat{\mathbb{C}} \setminus U$ contains at least two points, then there is a biholomorphic mapping $\varphi : U \to \mathbb{D}$. Furthermore, if $p \in U$ then φ can be chosen such that $\varphi(p) = 0$ and is then uniquely determined up to rotations.

Let $\chi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the following biholomorphic function

$$\chi(z) = \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C} \setminus \{0\} \\ 0 & \text{if } z = \infty \\ \infty & \text{if } z = 0. \end{cases}$$

Assume that $K \subset \mathbb{C}$ is a connected and simply connected compact set and let $U = \hat{\mathbb{C}} \setminus K$. Then U is a simply connected open set in $\hat{\mathbb{C}}$ containing ∞ and by the Riemann mapping theorem there is a biholomorphic function $\varphi : U \to \mathbb{D}$ such that $\varphi(\infty) = 0$. Define $\psi : \hat{\mathbb{C}} \setminus K \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, $\psi = \chi \circ \varphi$ and note that $\psi(\infty) = \infty$. Since φ is biholomorphic, $\varphi'(\infty) \neq 0$, that is $|\frac{1}{\varphi(z)}/z|$ is bounded when $|z| \to \infty$. This results in $|\psi(z)| = O(|z|)$ as $|z| \to \infty$. Then $\log|\psi|$ is harmonic and $\log|\psi| \sim \log|z|$. The function ψ is continuous and proper, so $|\psi(z)| \to 1$ when $z \to \partial K$. We have then derived a solution of the Dirichlet problem, namely $g_K = \log|\psi|$.

Let's look at the specific example when K is the interval $[-1, 1]$ on the real axis. Then the function ψ above is given by $z \mapsto z + \sqrt{z^2 - 1}$, from $\mathbb{C} \setminus K$ to $\mathbb{C} \setminus \overline{\mathbb{D}}$. It is the inverse of the Joukowski transform, $w \mapsto \frac{1}{2}(w + \frac{1}{w})$, which maps both $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ onto $\mathbb{C} \setminus K$. The solution to the Dirichlet problem is then $g_K = \log|z + \sqrt{z^2 - 1}|$.

For a detailed discussion about these subjects see for example Krantz [9], Kellog [7], and Nehari [17].

1.1.3 Walsh approximation theorem

We now wish to tie together the Dirichlet problem discussed above and extensions of holomorphic functions. This is done by looking at the sublevel set of the solution to the Dirichlet problem. We will denote the sublevel sets as $\Omega_R = \{z \in \mathbb{C} \setminus K : g_K(z) < \log R\}$ and their boundary $K_R = \{z \in \mathbb{C} \setminus K : g_K(z) = \log R\}$.

This result is due to Walsh and can be found in his detailed book about approximations and interpolation in $\mathbb{C}$, [24].

Theorem 5 in [24], §4.5: Let K be a compact set of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected and the Dirichlet problem for $\mathbb{C} \setminus K$ is solvable. If f is holomorphic in a neighbourhood of $\overline{\Omega}_R$, then there exist polynomials P_d of degrees $\leq d$ and a constant M such that

$$\|f - P_d\|_K \leq \frac{M}{R^d}.$$

There is also a kind of converse to this theorem:

Theorem 6 in [24], §4.6: Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected and the Dirichlet problem for $\mathbb{C} \setminus K$ is solvable. Assume f is holomorphic in a neighbourhood of K and there are polynomials P_d of degree $\leq d$, a constant M and a $R > 1$ such that

$$\|f - P_d\|_K \leq \frac{M}{R^d}, \qquad \text{for all } d \geq 1.$$

The sequence P_d is then uniformly convergent on all relatively compact subsets of Ω_R and f then extends to a holomorphic function on Ω_R.

When we look at the solution $\log|\psi|$ to the Dirichlet problem obtained by the Riemann mapping theorem in the previous section we see that $K_R = \{z : |\psi(z)| = R\} = \psi^{-1}(\partial B(0,R))$. In the example above where $K = [-1,1]$, ψ was given as the inverse of the Joukowski transformation. Then K_R is the image of the circle with radius R and center 0 under the Joukowski transformation. These curves are ellipses with foci ±1 and semiaxes $\frac{1}{2}(R + 1/R)$ and $\frac{1}{2}(R - 1/R)$.

1.1.4 Subharmonic functions and the Perron method

When looking at ways to generalize Walsh's theorem to higher dimensions we have to look at different methods than were used in Section 1.1.2, since the Riemann mapping theorem fails for $\mathbb{C}^n$, $n > 1$. This method is provided by the *Perron method*, ([3] §2.6), which is easily generealized to $\mathbb{C}^n$ as the *Perron-Bremermann method* ([8], page 89).

The Perron method is used to solve the Dirichlet problem using subharmonic functions. Subharmonic functions are those functions which satisfy a certain convexity property with respect to harmonic functions. Namely, an upper semicontinuous (see Definition 2.1.1) function u on an open subset Ω in $\mathbb{C}$ is subharmonic if the following is satisfied: For a compact set $K \subset \Omega$ and $h \in \mathcal{C}(K)$ harmonic inside of K, then $u \leq h$ on ∂K implies $u \leq h$ on K.

The reason for using subharmonic functions is that they are closely connected to harmonic function but more flexible and less fragile in some sense. Perron's method provides a solution to the Dirichlet problem in the following way:

Let Ω be a connected open set $\mathbb{C}$, and f a continuous function on the boundary, $\partial\Omega$, of Ω. We define the class $U(f)$ as all subharmonic functions u on Ω such that

$\limsup_{\Omega \ni z \to z_0} u(z) \leq f(z_0)$ for all $z_0 \in \partial\Omega$.

Under certain regularity conditions on $\partial\Omega$, the function v, given by

$$v(z) = \sup_{u \in U(f)} u(z), \qquad z \in \Omega,$$

is a solution of the Dirichlet problem, see Hayman and Kennedy [3] Theorem 2.11.

Note that harmonicity and subharmonicity are real-variable properties. They are however of importance in $\mathbb{C}$, which we can identify with $\mathbb{R}^2$. If f is holomorphic then $Re(f)$ and $Im(f)$ are harmonic, and conversely a harmonic function is locally the real part of a holomorphic function. Furthermore, subharmonicity is invariant under composition with holomorphic functions on connected domains.

When in $\mathbb{C}^n$, $n > 1$ things get harder. For example, subharmonicity is not invariant under holomorphic composition and there are harmonic functions which are not the real part of a holomorphic function. The class of subharmonic functions is therefore too large when $n > 1$. The solution is to introduce the plurisubharmonic functions, which are the functions that are subharmonic along every complex line. In $\mathbb{C}$ the plurisubharmonic functions coincide with the subharmonic functions and when $n > 1$ they are a subclass of the subharmonic functions. The plurisubharmonic functions are therefore a natural extension of the subharmonic functions to higher complex dimensions and they are the functions we will use when introducing the *pluricomplex Green function*. This function will play the role of g_K in *Siciak's approximation theorem* in Chapter 3, which is a generalization of Walsh's theorem to $\mathbb{C}^n$.

2

The pluricomplex Green function

This chapter is devoted to the pluricomplex Green functions and their properties. We start by introducing plurisubharmonic functions and a special class of them, called the Lelong class. We study some properties of the Lelong class and then use it to define the pluricomplex Green function $V_{X,q}$, for a set $X \subset \mathbb{C}^n$ and a weight function q on X. The results about plurisubharmonic functions and the Lelong class then enable us to acquire some information about the continuity of the Green function with respect to the set and weight in question.

2.1 Plurisubharmonicity

2.1.1 Semicontinuity and subharmonicity

Although there are similarities between convex functions and subharmonic functions, there is one fundamental difference, subharmonic functions do not need to be continuous. They are however upper semicontinuous, a weaker condition than continuity. Here will we look at the most important properties of upper semicontinuous and subharmonic functions needed for our study of plurisubharmonic functions.

Definition 2.1.1 Let (Ω, d) be a metric space. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be *upper semicontinuous* if $\{z \in \Omega : u(z) < r\}$ is open for every $r \in \mathbb{R}$. A function u is *lower semicontinuous* if $-u$ is upper semicontinuous.

Upper semicontinuity of u is equivalent to $\limsup_{z\to z_0} u(z) = u(z_0)$ for every $z_0 \in \Omega$, where $\limsup_{z\to z_0} u(z) = \inf_{\varepsilon>0} \big\{ \sup\{u(z) : z \in \Omega, d(z, z_0) < \varepsilon\}\big\}$, that is for every $\alpha > u(z_0)$ there is an $\varepsilon > 0$ such that $u(z) < \alpha$ if $d(z, z_0) < \varepsilon$.

Note that a real valued function is continuous if and only if it is both lower- and upper semicontinuous.

Definition 2.1.2 For $\Omega \subset \mathbb{C}^n$ let $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be a function which is locally bounded from above. Then u^* defined on Ω by $u^*(z_0) = \limsup_{z \to z_0} u(z)$ is called the *upper semicontinuous regularization of* u.

The function u^* is upper semicontinuous, moreover $u \leq u^*$ and if v is an upper semicontinuous function such that $u \leq v$, then $u^* \leq v$.

Proposition 2.1.3 (Prop. 2.3.1 in [8]) *If* u *is an upper semicontinuous function on a compact set* K*, then* u *is bounded above and there is a point where it attains its maximum.*

Another well known property of upper semicontinuous functions is the following.

Proposition 2.1.4 (Prop. 2.3.3 in [8]) *If* u *is an upper semicontinuous function on a compact set* K*, then there is a decreasing sequence of continuous functions* $\{u_j\}$ *converging to* u*.*

Remember that a function $u \in \mathcal{C}^2(\Omega)$ on an open set $\Omega \subset \mathbb{C}$ is called *harmonic* if

$$\Delta u(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 4\frac{\partial^2 u}{\partial z \partial \bar{z}}(z) = 0, \qquad z = x + iy \in \Omega.$$

Definition 2.1.5 Let $\Omega \subset \mathbb{C}$ be open and $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. The function u is said to be *subharmonic* if for every compact set $K \subset \Omega$ and every function $h \in \mathcal{C}(K)$, harmonic on the interior of K, the following holds:

$$\text{if } u \leq h \text{ on } \partial K \text{ then } u \leq h \text{ on } K.$$

Obviously every harmonic function is subharmonic.

Although this definition is lucid and describes the convexity property of subharmonic functions it is not always very usefull in the context of real and complex analysis. But as harmonic functions are characterized by the *mean value property*, the subharmonic functions are characterized by the *sub-mean value property.*

Proposition 2.1.6 (Th. 2.1.4 in [10]) *Let u be an upper semicontinuous function on $\Omega \subset \mathbb{C}$. Then u is subharmonic on Ω if and only if*

$$u(a) \leq \frac{1}{2\pi r} \int_{\partial B(a,r)} u(z)\, d\sigma(z)$$

holds for every $\overline{B}(a,r) \subset \Omega$, where σ is the arc length measure on $\partial B(a,r)$.

It is also possible to define subharmonicity using the Laplacian, but since a subharmonic function is not necessarily twice differentiable this has to be done in the sense of distributions.

Proposition 2.1.7 (Prop. 2.1.10 in [10] and Th. 2.5.8 in [8]) *Let $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function on an open set Ω, and assume u is not identically $-\infty$ on any connected component of Ω. Then u is locally integrable, $u^{-1}(-\infty)$ has Lebesgue measure zero and $\Delta u \geq 0$ in the sense of distributions, i.e.,*

$$\int_{\Omega} u(z) \Delta\varphi(z)\, d\lambda(z) \geq 0$$

for all non-negative test functions $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

Conversely, if u is locally integrable and $\Delta u \geq 0$ in the sense of distributions, then there is a subharmonic function equal to u almost everywhere. Here λ denotes the Lebesgue measure on $\mathbb{C}$.

Note, if $u \in \mathcal{C}^2(\Omega)$ is subharmonic, then $\Delta u \geq 0$ means that

$$\Delta u(z) = 4 \frac{\partial^2}{\partial z \partial \bar{z}} u(z) \geq 0, \qquad z \in \Omega.$$

The following provides a kind of converse of Proposition 2.1.6.

Proposition 2.1.8 (Th. 2.5.9 in [8]) *Let u be a locally integrable function on an open set $\Omega \subset \mathbb{C}$ such that*

$$u(a) \leq \frac{1}{2\pi r} \int_{\partial B(a,r)} u(z)\, d\sigma(z)$$

for all $\overline{B}(a,r) \subset \Omega$. Then there is an unique subharmonic function on Ω equal to u almost everywhere.

We have shown here three equivalent conditions for subharmonic functions, convexity with respect to harmonic functions, the submean-value property (Proposition 2.1.6) and that the Laplacian is non-negative in the sense of distributions (Theorem 2.1.7).

Finally, we present *Hartogs lemma*, which can be of great value and provides a key step in proving Hartogs' theorem about separate analyticity.

Theorem 2.1.9 (Th. 2.6.4 in [8]) *Let* $\{u_j\}_j$ *be a sequence of subharmonic functions on an open set* $\Omega \subset \mathbb{C}$ *which is locally uniformly bounded above. Assume there is a constant* M *such that*

$$\limsup_{j\to\infty} u_j(z) \le M, \qquad z \in \Omega.$$

Then for every $\varepsilon > 0$ *and every compact set* $K \subset \Omega$, *there is a* $j_0 \in \mathbb{N}$ *such that for* $j \ge j_0$

$$\|u_j\|_K \le M + \varepsilon.$$

2.1.2 Plurisubharmonicity

Here we list the properties of plurisubharmonic functions needed for our studies of the pluricomplex Green function. The proofs and a detailed survey of the theory of plurisubharmonic functions can be found in Klimek, [8].

Definition 2.1.10 Let $\Omega \subset \mathbb{C}^n$ be open and $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. The function u is said to be *plurisubharmonic* if for every $\alpha, \beta \in \mathbb{C}^n$, the function

$$\zeta \mapsto u(\alpha + \zeta\beta),$$

is subharmonic on $\{\zeta \in \mathbb{C} : \alpha + \zeta\beta \in \Omega\}$. The family of plurisubharmonic functions on an open set Ω will be noted by $\mathcal{PSH}(\Omega)$.

The set of plurisubharmonic functions is a subset of the subharmonic functions, looking at subharmonicity in $\mathbb{C}^n$ as in $\mathbb{R}^{2n}$, coinciding only if $n = 1$.

The functions $\log|f|$ and $|f|^p$, $p > 0$, where $f : \Omega \to \mathbb{C}^m$ is a holomorphic function, are examples of plurisubharmonic functions. It is also obvious that both the sum and the maximum of a finite number of plurisubharmonic functions are plurisubharmonic and that plurisubharmonicity is invariant under multiplication with a positive constant. If $f_1, \ldots, f_p$ are holomorphic functions and $\gamma_1, \ldots, \gamma_p$ are positive constants, then

$$\log(|f_1|^{\gamma_1} + \cdots + |f_p|^{\gamma_p})$$

is plurisubharmonic. Also, if $f : \Omega \to \mathbb{C}^m$ is a holomorphic map and u is plurisubharmonic on the image of f then $u \circ f$ is plurisubharmonic.

Another class of plurisubharmonic functions are the following. Let $\omega \subset \mathbb{R}^n$ be an open set and $\Omega = \omega \oplus i\mathbb{R}^n$, a tube domain. Assume $\varphi(x + iy) = \varphi(x)$ is a function depending only on $x \in \omega$. Then the function $z \mapsto \varphi(z)$ is plurisubharmonic on Ω if and only if $x \mapsto \varphi(x)$ is convex on ω.

The following properties are consequences of corresponding properties for subharmonic functions.

Proposition 2.1.11 (Cor. 2.9.9 in [8]) *If u is plurisubharmonic on an open connected set Ω and there is a $z \in \Omega$ such that $u(z) = \sup_\Omega u$, then u is constant on Ω.*

Proposition 2.1.12 (Cor. 2.2.6 in [10]) *If u is plurisubharmonic on Ω and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and monotonically increasing, then $\varphi \circ u$ is plurisubharmonic (here $\varphi(-\infty)$ is understood as $\lim_{t\to-\infty} \varphi(t)$).*

Proposition 2.1.13 (Th. 2.9.14(ii) in [8]) *If $\{u_j\}_j$ is a decreasing sequence in $\mathcal{PSH}(\Omega)$, then $\lim_{j\to\infty} u_j$ is in $\mathcal{PSH}(\Omega)$.*

Proposition 2.1.14 (Th. 2.9.14(iv) in [8]) *Let $\mathcal{F} \subset \mathcal{PSH}(\Omega)$ be a family of plurisubharmonic functions which is locally bounded above and $u(z) = \sup_{u\in\mathcal{F}} u(z)$. Then $u^* \in \mathcal{PSH}(\Omega)$.*

If $u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^2(\Omega)$ then it follows from Proposition 2.1.7 that

$$\frac{1}{4}\Delta_\zeta u(z + \zeta w)|_{\zeta=0} = \frac{\partial^2}{\partial\zeta\partial\overline{\zeta}} u(z + \zeta w)|_{\zeta=0} = \sum_{j,k=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} u(z) w_j \overline{w}_k \geq 0$$

for all $z \in \Omega$ and $w \in \mathbb{C}^n$. Proposition 2.1.7 has the following generalization to higher dimensions.

Proposition 2.1.15 (Cor. 2.9.10 in [8]) *If a function u is plurisubharmonic on an open set Ω and not identically $-\infty$ on any connected component of Ω, then u is* *locally integrable,* $u^{-1}(-\infty)$ *has Lebesgue measure zero, and*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \, \partial \overline{z}_k} w_j \overline{w}_k \geq 0, \qquad w \in \mathbb{C}^n,$$

in the sense of distributions, i.e.,

$$\int_\Omega u \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \, \partial \overline{z}_k} w_j \overline{w}_k \, d\lambda \geq 0,$$

for all non-negative test functions $\varphi \in \mathbb{C}_c^\infty(\Omega)$*, and all* $w \in \mathbb{C}^n$*.*

Conversely, if v *is a real valued distribution on* Ω *such that*

$$\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \, \partial \overline{z}_k} w_j \overline{w}_k \geq 0, \qquad w \in \mathbb{C}^n,$$

in the sense of distributions, then there is an unique plurisubharmonic function u *representing* v*, i.e.,* $v(\varphi) = \int u\varphi \, d\lambda$ *for all test functions* $\varphi \in \mathcal{C}_c^\infty(\Omega)$*. Here* λ *denotes the Lebesgue measure on* $\mathbb{C}$*.*

Convolution is a standard operation for regularizing functions and distributions and it works well for plurisubharmonic functions.

If $\Omega \subset \mathbb{C}^n$ is open and $u \in \mathcal{PSH}(\Omega)$, let ω be a non-negative function in $\mathcal{C}_c^\infty(\mathbb{C}^n)$ with support in the unit ball. Set $\omega_\delta(z) = \delta^{-2n}\,\omega(\delta^{-1}z)$ for $\delta > 0$. If ω is radially symmetric, i.e., if ω only depends on $|z|$ and $\int_{\mathbb{C}^n} \omega \, d\lambda = 1$, then

$$u * \omega_\delta(z) = \int_\Omega u(z-y)\omega_\delta(y)d\lambda(y) = \int_\Omega u(y)\omega_\delta(z-y)d\lambda(y)$$

is well defined for all $z \in \Omega_\delta$, where $\Omega_\delta = \{z \in \Omega : d(z, \Omega^c) > \delta\}$. Furthermore, $u * \omega_\delta \in \mathcal{PSH}(\Omega_\delta) \cap \mathcal{C}^\infty(\Omega_\delta)$ and for every $z \in \Omega$ the function $\delta \mapsto u * \omega_\delta(z)$ is increasing in the open interval where it is defined and $u(z) = \lim_{\delta \to 0} u * \omega_\delta(z)$. If we take a decreasing sequence $\delta_j \searrow 0$ we get the following.

Proposition 2.1.16 (Th. 2.9.2 in [8]) *Let* Ω *be an open subset of* $\mathbb{C}^n$ *and* u *a plurisubharmonic function on* Ω*, then for every* $\delta_0 > 0$*, there exists a decreasing sequence* $\{u_j\}_j$ *in* $\mathcal{PSH}(\Omega_{\delta_0}) \cap \mathcal{C}^\infty(\Omega_{\delta_0})$ *such that* $u_j(z) \to u(z)$ *for all* $z \in \Omega_{\delta_0}$*.*

If $\Omega = \mathbb{C}^n$, then of course $\Omega_\delta = \mathbb{C}^n$ for every $\delta > 0$.

Definition 2.1.17 A set $E \subset \mathbb{C}^n$ is called *locally pluripolar* if each point $a \in E$ has an open neighbourhood V and there is a plurisubharmonic function v on V such that $E \cap V \subset \{z \in V : v(z) = -\infty\}$. A set $E \subset \mathbb{C}^n$ is called *(globally) pluripolar* if there is a function $v \in \mathcal{PSH}(\mathbb{C}^n)$ such that $E \subset \{z \in \mathbb{C}^n : v(z) = -\infty\}$.

Josefson [6], showed that locally pluripolar sets are globally pluripolar. Obviously globally pluripolar sets are locally pluripolar, hence these conditions are equivalent. Good examples of pluripolar sets are the zero sets of holomorphic functions, because $\log |f|$ is plurisubharmonic if f is holomorphic.

Theorem 2.1.18 (Th. 2.9.22 in [8]) *Let $\Omega \subset \mathbb{C}^n$ be open, $F \subset \Omega$ closed, and assume there is a function $v \in \mathcal{PSH}(\Omega)$ such that $F \subset \{z \in \Omega : v(z) = -\infty\}$. Then every $u \in \mathcal{PSH}(\Omega \setminus F)$, which is locally bounded above near each point of F, can be extended to $\tilde{u} \in \mathcal{PSH}(\Omega)$, and $\tilde{u}$ is given by the following formula*

$$\tilde{u}(z) = \begin{cases} u(z), & z \in \Omega \setminus F, \\ \limsup_{\Omega \setminus F \ni y \to z} u(y), & z \in F. \end{cases}$$

2.1.3 Polynomials, homogeneity and balanced sets

This section contains some results regarding polynomials, homogeneous and holomorphic functions, and also their connection with balanced sets. The results here will play a vital role in Chapter 3.

Definition 2.1.19 A set $E \subset \mathbb{C}^n$ is called *n-circular* if for every $a \in E$, all points $z \in \mathbb{C}^n$ such that $|a| = |z|$ are in E, *circular* if for every $a \in E$, $\lambda a \in E$ for all $\lambda \in \mathbb{T}$, and *balanced* if for every $a \in E$, $\lambda a \in E$ for all $\lambda \in \overline{\mathbb{D}}$.

We will denote the polynomial hull of a compact set K by

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K \text{ for all polynomials } p\}.$$

Lemma 2.1.20 *If K is a balanced compact subset of $\mathbb{C}^n$, then*

$$\hat{K} = \{z \in \mathbb{C}^n : |Q(z)| \leq \|Q\|_K \textit{ for all homogeneous polynomials } Q \textit{ on } \mathbb{C}^n\}.$$

Proof: For proving that the right hand side is included in $\hat{K}$ we take b in the right hand side and a non-constant polynomial P. Then we can write $P = \sum_{j=0}^{d} Q_j$, Q_j

a homogeneous polynomials of degree j or identically zero. For every $z \in K$ we apply the Cauchy estimate $|D^j f(a)| \leq j!\|f\|_{\partial B(a,r)}/r^j$ to the holomorphic function $\lambda \mapsto P(\lambda z) = \sum_{j=0}^d Q_j(\lambda z) = \sum_{j=0}^d \lambda^j Q_j(z)$ with $a = 0$ and $r = 1$. It yields $|Q_j(z)| \leq \|P\|_{\{\lambda z : \lambda \in \mathbb{T}\}} \leq \|P\|_K$ since K is balanced. Since this holds for every $z \in K$ it implies $|P(b)| \leq \sum_{j=0}^d |Q_j(b)| \leq \sum_{j=0}^d \|Q_j\|_K \leq (d+1)\|P\|_K$, and consequently

$$|P(b)|^{1/d} \leq (d+1)^{1/d}\|P\|_K^{1/d}.$$

Applying the same argument to P^k, $k = 1, 2, \dots$ shows that

$$|P(b)|^{1/d} = |P^k(b)|^{1/kd} \leq (kd+1)^{1/kd}\|P^k\|_K^{1/kd} = (kd+1)^{1/kd}\|P\|_K^{1/d},$$

and therefore $|P(b)| \leq \lim_{k\to\infty}(kd+1)^{1/k}\|P\|_K = \|P\|_K$ showing $b \in \hat{K}$. The other inclusion is obvious from the definition of the polynomial hull $\hat{K}$ of K. □

Theorem 2.1.21 *Let Ω be a balanced open neighbourhood of the origin in $\mathbb{C}^n$ and f a holomorphic function on Ω, then there are homogeneous polynomials P_j of degree j such that*

$$f = \sum_{j=0}^{\infty} P_j,$$

and the series is uniformly convergent on compact subsets of Ω.

Proof: Find $r > 0$ such that the polydisc $\overline{P}(0, r)$ is in Ω and expand f in a power series

$$f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha,$$

which is absolutely and uniformly convergent on $\overline{P}(r, 0)$. Define for $j \in \mathbb{Z}_+$

$$P_j(z) = \sum_{|\alpha|=j} a_\alpha z^\alpha.$$

Then P_j are homogeneous polynomials on $\mathbb{C}^n$ of degree j.

For a compact subset K of Ω, let $t > 1$ be number such that $t^2K \subset \Omega$ and define plurisubharmonic functions $u_j = |P_j|^{1/j}$ on $\mathbb{C}^n$. By uniform convergence on $\overline{P}(r, 0)$ there is a number $M > 0$ such that $\|P_j\|_{\overline{P}(0,r)} \leq M$ for all $j \geq 0$, so for all j we have

$$u_j(z) \leq M^{1/j}\frac{|z|}{r}, \qquad z \in \mathbb{C}^n,$$

and the family $\{u_j\}$ is locally uniformly bounded above. For $z \in \Omega$, find $s > 1$ such that the set $\{\lambda z : \lambda \in D(0,s)\}$ is relatively compact in Ω, this is possible since Ω is balanced. The function $\lambda \mapsto f(\lambda z)$ is then holomorphic on $D(0,s)$, and thus the series

$$f(\lambda z) = \sum_{j=0}^{\infty} P_j(\lambda z) = \sum_{j=0}^{\infty} P_j(z)\lambda^j,$$

is absolutely convergent for $|\lambda| < s$. This means that the radius of convergence of this power series in λ is greater or equal to 1 and we get

$$\limsup_{j\to\infty} u_j(z) = \limsup_{j\to\infty} |P_j(z)|^{1/j} \leq 1$$

for all $z \in \Omega$. By Hartogs lemma, (Theorem 2.1.9), there is a j_0 such that if $j \geq j_0$ then

$$u_j(z) \leq t, \qquad z \in t^2 K.$$

Then for $z \in K$ and $j \geq j_0$ we have $|P_j(z)| = t^{-2j}|P_j(t^2 z)| \leq t^{-j}$, therefore the series $\sum P_j$ is uniformly convergent on K. ☐

Definition 2.1.22 Let $\mathcal{H}_+^n$ be the family of all functions $u \in \mathcal{PSH}(\mathbb{C}^n)$ which are non-negative and complex homogeneous of degree 1, i.e., $u(\lambda z) = |\lambda| u(z)$ for $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$, and not identically zero.

We have seen how convolution can be used to regularize plurisubharmonic functions. Now we present a related method for regularizing homogeneous functions.

Theorem 2.1.23 *If $u \in \mathcal{H}_+^n$ there is a family $\{u_\delta\}_{\delta>0}$ of continuous functions in $\mathcal{H}_+^n \cap \mathcal{C}^\infty(\mathbb{C}^n \setminus \{0\})$, such that $u_\delta \searrow u$ as $\delta \to 0$.*

Proof: Let $\mathbb{C}^{n\times n}$ be the space of all complex $n \times n$ matrices and I the identity matrix in $\mathbb{C}^{n\times n}$. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{C}^{n\times n})$ be radially symmetric, non-negative and such that $\int_{\mathbb{C}^{n\times n}} \varphi \, d\lambda = 1$. For $\delta > 0$ set $\varphi_\delta(A) = \delta^{-2n^2}\varphi((A-I)/\delta)$ and define

$$u_\delta(z) = \int_{\mathbb{C}^{n\times n}} u(Az)\varphi_\delta(A)\, d\lambda(A), \qquad z \in \mathbb{C}^n. \tag{2.1}$$

The plurisubharmonicity of u_δ follows by substituting the integral in (2.1) into

$$u_\delta(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u_\delta(a + b\,e^{i\theta})\, d\theta, \qquad a, b \in \mathbb{C}^n$$

and then change the order of integration. Note that u_δ is homogeneous and therefore continuous at zero. If $U \in L^1_{\text{loc}}(\mathbb{C}^{n\times n})$, then we define a function U_δ as before by

$$U_\delta(Z) = \int_{\mathbb{C}^{n\times n}} U(AZ)\,\varphi_\delta(A)\,d\lambda(A), \qquad Z \in \mathbb{C}^{n\times n}.$$

If $Z \in GL_n(\mathbb{C})$, then the real determinant of the linear map $A \mapsto AZ$ is $|\text{det}Z|^{2n}$, so a change of variable $B = AZ$ gives

$$U_\delta(Z) = \int_{\mathbb{C}^{n\times n}} U(B)\,\varphi_\delta(BZ^{-1})|\text{det}Z|^{-2n}\,d\lambda(B).$$

The function $GL_n(\mathbb{C}) \ni Z \mapsto \varphi_\delta(BZ^{-1})|\text{det}Z|^{-n}$ is in $\mathcal{C}^\infty(GL_n(\mathbb{C}))$ and it follows that $U_\delta \in \mathcal{C}^\infty(GL_n(\mathbb{C}))$.

Our function u defines a function U in $L^1_{loc}(\mathbb{C}^{n\times n})$ by $U(Z) = u(Z_1)$, where Z_1 is the first column of the matrix Z. Clearly, $U_\delta(Z) = u_\delta(Z_1)$ and therefore $u_\delta \in \mathcal{C}^\infty(\mathbb{C}^n \setminus \{0\})$.

To show that $u_\delta \searrow u$ when $\delta \to 0$, we change the variable A in formula (2.1) to $B = (A - I)/\delta$, then

$$u_\delta(z) = \int_{\mathbb{C}^{n\times n}} u(z + \delta Bz)\,\varphi(B)\,d\lambda(B).$$

Since φ is radially symmetric there exists $\psi \in \mathcal{C}^\infty_0(\mathbb{R})$ such that $\varphi(B) = \psi(|B|)$. Then, using polar coordinates,

$$u_\delta(z) = \int_0^\infty \left(\int_S u(z + \delta r\omega z)\,d\omega\right)\psi(r)\,dr,$$

where S is the unit sphere in $\mathbb{C}^{n\times n}$. The sub-mean value property of u shows that the inner integral is an increasing functions of δ, see [10], Lemma 2.1.17. By taking the limit under the integrals it follows that $u_\delta \searrow u$, as $\delta \to 0$. ☐

2.2 The Lelong class

We start by defining the *Lelong class* of plurisubharmonic functions. This class is the cornerstone for the studies of the pluricomplex Green functions, in the same sense as polynomials are for Siciak's extremal function $\Phi_{K,q}$ [21].

Definition 2.2.1 A function $u \in \mathcal{PSH}(\mathbb{C}^n)$ is said to be of *logarithmic growth* if there is a constant C such that

$$u(z) \le \log^+ |z| + C, \qquad z \in \mathbb{C}^n,$$

where $\log^+ x = \max\{0, \log x\}$. The family of all such functions is called the *Lelong-class* and denoted by $\mathcal{L}$.

Every function of the form $\frac{1}{\deg P} \log |P|$ where P is a polynomial on $\mathbb{C}^n$ is in $\mathcal{L}$.

It is convenient in Chapters 3 and 4 to have special notation for points in $\mathbb{C}^{n+1} = \mathbb{C} \times \mathbb{C}^n$, namely, we write $\tilde{z} = (z_0, z)$, where $z_0 \in \mathbb{C}$ and $z \in \mathbb{C}^n$.

In some cases we identify $\mathbb{C}^n$ with the hyperplane $\{1\} \times \mathbb{C}^n \subset \mathbb{C}^{n+1}$ and view a function $\tilde{v}$ on $\mathbb{C}^{n+1}$ as an extension of a function v on $\mathbb{C}^n$ if $\tilde{v}(1, z) = v(z)$ for all $z \in \mathbb{C}^n$.

As mentioned in Chapter 1, the complex projective space $\mathbb{P}^n$ is defined as the space of all complex lines in $\mathbb{C}^{n+1}$. We define the natural projection $(z_0, \dots, z_n) \mapsto [z_0 : \cdots : z_n]$ by $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. We identify $\mathbb{C}^n$ with the subspace of $\mathbb{P}^n$ consisting of all $[z_0 : \cdots : z_n]$ with $z_0 \neq 0$, and we write the values of the projection π as $\pi(z_0, z_1, \dots, z_n) = (z_1/z_0, \dots, z_n/z_0)$. The hyperplane at infinity H_∞ in $\mathbb{P}^n$ is then the projection of the set $\{\tilde{z} = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} : z_0 = 0\}$. We can therefore view $\mathbb{P}^n$ as the union of $\mathbb{C}^n$ and H_∞.

Proposition 2.2.2 *A function $u \in \mathcal{PSH}(\mathbb{C}^n)$ is in the Lelong class $\mathcal{L}$ if and only if the function*

$$\tilde{z} = (z_0, \dots, z_n) \mapsto u \circ \pi(\tilde{z}) + \log |z_0| = u(z_1/z_0, \dots, z_n/z_0) + \log |z_0| \tag{2.2}$$

extends as a plurisubharmonic function from $\mathbb{C}^{n+1} \setminus \{\tilde{z} : z_0 = 0\}$ to $\mathbb{C}^{n+1} \setminus \{0\}$.

Proof: If $u \in \mathcal{L}$ and we let $\varphi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R} \cup \{-\infty\}$ denote the function $\tilde{z} \mapsto \log |z_0|$ then $u \circ \pi + \varphi$ is obviously plurisubharmonic on $\mathbb{C}^{n+1} \setminus \{\tilde{z} : z_0 = 0\}$, and

$$\begin{aligned}
u \circ \pi(z) + \varphi(z) &= u(z_1/z_0, \dots, z_n/z_0) + \log |z_0| \\
&\le \log^+ |(z_1/z_0, \dots, z_n/z_0)| + \log |z_0| + C \\
&= \log^+ (\frac{1}{|z_0|} |(z_1, \dots, z_n)|) + \log |z_0| + C \\
&\le \log^+ |(z_1, \dots, z_n)| + C
\end{aligned}$$

so $u \circ \pi + \varphi$ is locally bounded above near each point of the form $(0, w)$, $w \in \mathbb{C}^n \setminus \{0\}$ and extends therefore to a plurisubharmonic function on $\mathbb{C}^{n+1} \setminus \{0\}$ by Theorem 2.1.18.

If $u \circ \pi + \varphi$ extends to a plurisubharmonic function on $\mathbb{C}^{n+1} \setminus \{0\}$ then $z \mapsto u(z) = u \circ \pi(1, z) + \varphi(1)$ is plurisubharmonic on $\mathbb{C}^n$. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n \setminus \{0\}$ let $z_0 = \frac{1}{|z|}$, then $u \circ \pi(z_0, \frac{z}{|z|}) + \log |z_0|$ is locally bounded above.

Let $K = \overline{B}(0, \varepsilon) \times S^n$ be a compact subset in $\mathbb{C}^n$ where $\varepsilon > 0$ and C are chosen such that $u \circ \pi(z_0, \frac{z}{|z|}) + \log |z_0| \leq C$ on K, that is

$$u(z) \leq \log^+ |z| + C, \qquad \text{for } |z| \text{ large enough .}$$

□

As for plurisubharmonic functions, we define the polar set for $\mathcal{L}$.

Definition 2.2.3 A set $E \subset \mathbb{C}^n$ is called *$\mathcal{L}$-polar* if there exists $u \in \mathcal{L}$ such that $u|_E = -\infty$.

Proposition 2.2.4 (Prop. 5.2.1 in [8]) *Let $\mathcal{U} \subset \mathcal{L}$ be a non-empty family and $u(z) = \sup\{v(z) : v \in \mathcal{U}\}$, a function on $\mathbb{C}^n$. If $\{z \in \mathbb{C}^n : u(z) < +\infty\}$ is not $\mathcal{L}$-polar then the family $\mathcal{U}$ is locally bounded above and $u^* \in \mathcal{L}$.*

Proposition 2.2.5 (Prop. 5.2.3 in [8]) *If $E := \cup_{j=1}^{\infty} E_j$ is the union of countably many $\mathcal{L}$-polar sets then E is $\mathcal{L}$-polar.*

Theorem 2.2.6 (Th. 5.2.4 in [8]) *Pluripolar sets are $\mathcal{L}$-polar.*

As noted previously, Josefson [6] showed that locally pluripolar sets are indeed globally pluripolar. This, along with the theorem above and the obvious observation that $\mathcal{L}$-polar sets are globally pluripolar shows that the following properties are equivalent for a set $E \subset \mathbb{C}^n$,

(i) E is locally pluripolar,

(ii) E is globally pluripolar,

(iii) E is $\mathcal{L}$-polar.

As with plurisubharmonic and homogeneous functions we wish to able to approximate functions from the Lelong class with smooth functions from the same class.

Theorem 2.2.7 *If $u \in \mathcal{L}$ there is a decreasing family $\{u_\delta\}_{\delta>0}$ of $\mathcal{C}^\infty$ plurisubharmonic functions in $\mathcal{L}$ such that $u_\delta \searrow u$.*

Proof: We define ω_δ and u_δ as before the statement of Proposition 2.1.16. Then we only have to show that $u_\delta \in \mathcal{L}$. We have $\operatorname{supp}(\omega_\delta) \subset B(0,\delta)$ and $\int_{B(z,\delta)} \omega_\delta \, d\lambda = 1$. Hence,

$$\int_{\mathbb{C}^n} u(z-y)\omega_\delta(y)\, d\lambda(y) \leq \sup_{y \in B(z,\delta)} u(y) \leq \sup_{y \in B(z,\delta)} \log^+|y| + C \leq \log^+|z| + \delta + C.$$

$\square$

The next theorem can be found in Klimek [8] Theorem 5.1.6.

Theorem 2.2.8 *Let $h : \mathbb{C}^n \to [0,+\infty[$ be a function not identically 0 and $u : \mathbb{C}^n \to [-\infty,+\infty[$ a function not identically $-\infty$. Then*

(i) If $h \in \mathcal{H}_+^n$ is continuous and $h^{-1}(0) = \{0\}$, then

$$h(z) = \sup\{|Q(z)|^{1/\deg Q}\}, \qquad z \in \mathbb{C}^n, \tag{2.3}$$

where the supremum is taken over all complex homogeneous polynomials Q such that $|Q|^{1/\deg Q} \leq h$ in $\mathbb{C}^n$.

(ii) The function h is in $\mathcal{H}_+^n$ if and only if

$$h = \left(\limsup_{j\to\infty} |Q_j|^{1/j} \right)^*$$

for some sequence $\{Q_j\}$ of complex homogeneous polynomials, $\deg Q_j = j$. Consequently, if $h \in \mathcal{H}_+^n$, then $\log h \in \mathcal{L}$ by Proposition 2.2.4.

(iii) The function u is in $\mathcal{L}$ if and only if

$$e^u = \left(\limsup_{j\to\infty} |P_j|^{1/j} \right)^*$$

for some sequence $\{P_j\}$ of complex polynomials such that $\deg P_j \leq j$.

Proof: (i) Set $M := \inf\{h(z) : z \in \partial\mathbb{B}\}$, then M is nonzero since $h^{-1}(0) = \{0\}$ and h is continuous, hence $h(z) \geq M|z|$ for all $z \in \mathbb{C}^n$. Define g as the right hand side of (2.3). It is sufficient to show that $g(a) = 1$ for every a in $h^{-1}(1)$; since $h(z/h(z)) = 1$ for each $z \in \mathbb{C}^n \setminus \{0\}$, this implies $g(z/h(z)) = 1$ and hence, since g is also homogeneous, $h(z) = h(z)g(z/h(z)) = g(z)$.

Since h is homogeneous and plurisubharmonic, the set $D := \{z \in \mathbb{C}^n : h(z) < 1\}$ is a bounded and balanced domain of holomorphy. By Theorem 2.1.21 and the fact that D is holomorphically convex it is polynomially convex. Then for every $t \in \,]0, 1[$, $\hat{K}_t \subset\subset D$, where $K_t = h^{-1}([0, t])$.

Since $a \in h^{-1}(1)$ there is for $t \in]0, 1[$ a number $s \in]t, 1[$ such that $sa \in D \setminus \hat{K}_t$. Then by Lemma 2.1.20 there is a homogeneous polynomial Q such that $\|Q\|_{K_t} < Q(sa)$; by multiplying Q with a suitable constant we can assume

$$\|Q\|_{K_t} \leq 1 \leq Q(sa). \tag{2.4}$$

For $z \in \mathbb{C}^n$, $tz/h(z) \in K_t$ and hence $|Q(tz/h(z))|^{1/\deg Q} \leq 1$, that is $t\,|Q(z)|^{1/\deg Q} \leq h(z)$. Therefore, using (2.4) and the definition of g, $t \leq t|Q(sa)|^{1/\deg Q} \leq g(sa) = sg(a)$. Letting $t \to 1$, forces $s \to 1$ and the result is $1 \leq g(a)$. By definition of g we obviously have $g(a) \leq 1$.

(ii) If $h \in \mathcal{H}^n_+$ then the set $\Omega = \{w \in \mathbb{C}^n : h(w) < 1\}$ is a balanced domain of holomorphy since it is the sublevel set of a homogeneous plurisubharmonic function. Then there is a holomorphic function f on Ω which can not be extended to any point of $\partial\Omega$.

By Theorem 2.1.21, there are homogeneous polynomials Q_j such that $f = \sum Q_j$. Since the series is convergent on compact subsets of Ω then

$$\limsup_{j\to\infty} |Q_j(z)|^{1/j} < 1, \qquad \text{if } z \in \Omega,$$

and this holds only on Ω since f can not be extended to $\partial\Omega$.

Let

$$v(z) = \Big(\limsup_{j\to\infty} |Q_j(z)|^{1/j}\Big)^*, \qquad z \in \mathbb{C}^n,$$

then $\Omega = \{z \in \mathbb{C}^n : v(z) < 1\}$ and $v = h$ on $\mathbb{C}^n$, since v is non-negative and homogeneous, because if there is a $w \in \mathbb{C}^n$ such that $v(w) < h(w)$, choose an r such that $v(w) < r < h(w)$, then $v(w/r) < 1 < h(w/r)$ which contradicts the above, and the same goes for the case $v(w) > h(w)$. The other implication follows from (iii) when all the polynomials P_j are homogeneous and each of degree j.

(iii) For a function $u \in \mathcal{L}$, let $\tilde{v}$ be the extension of the function $\tilde{z} = (z_0, z) \mapsto u(z_1/z_0, \ldots, z_n/z_0) + \log|z_0|$ to $\mathbb{C}^{n+1} \setminus \{0\}$ from Proposition 2.2.2. Then $\tilde{h} = \exp(\tilde{v})$ is a homogeneous function, defined on $\mathbb{C}^{n+1}$ and $h(0) = 0$ by homogeneity. Then by (ii) there exists a sequence of homogeneous polynomials Q_j on $\mathbb{C}^{n+1}$ of degree j such that

$$h(1, z) = \exp\big(u(z)\big) = \Big(\limsup_{j\to\infty} |Q_j(1, z)|^{1/j}\Big)^*, \qquad z \in \mathbb{C}^n.$$

Setting $P_j(z) = Q_j(1, z)$ gives the desired result.

To show the opposite implication assume u satisfies the formula in (iii) for some polynomials P_j, $\deg P_j \leq j$. By the Uniform boundedness principle (page 299 in [16]), there is a ball $\overline{B} = \overline{B}(a, r) \subset \mathbb{C}^n$ and a positive constant M such that $\sup_{j\in\mathbb{N}} \|P_j\|_{\overline{B}}^{1/j} \leq M$. The set $\{z \in \mathbb{C}^n : \sup_{j\in\mathbb{N}} \frac{1}{j} \log|P_j(z)| < +\infty\}$ contains $\overline{B}$ and is therefore not $\mathcal{L}$-polar, then by Proposition 2.2.4 $u \in \mathcal{L}$. □

2.3 Pluricomplex Green function with logarithmic pole at infinity

Here we define the pluricomplex Green functions using the Lelong class $\mathcal{L}$ and a definition introduced in [25] by Zahariuta. We will derive its fundamental properties and calculate the Green function for the open ball.

Definition 2.3.1 For every $X \subset \mathbb{C}^n$ and $q : X \to [-\infty, +\infty[$ we define the *weighted pluricomplex Green function of X with weight q and logarithmic pole at infinity* by

$$V_{X,q}(z) = \sup\{u(z) : u \in \mathcal{L}, u \leq q \text{ on } X\}, \qquad z \in \mathbb{C}^n.$$

The important case when $q = 0$ is denoted by V_X.

The following is easily seen from the definition of $V_{X,q}$:

(i) $V_{A,q} \geq V_{B,q}$ if $A \subset B$,

(ii) $V_{X,q} \leq V_{X,p}$ if $q \leq p$,

(iii) $V_{X,q+c} = c + V_{X,q}$ for $c \in \mathbb{R}$.

From this we see that if q is bounded then

$$V_X + \inf_X q \leq V_{X,q} \leq V_X + \sup_X q.$$

Remark: Observe that if the set $q^{-1}(-\infty)$ is not $\mathcal{L}$-polar then $V_{X,q} = -\infty$. We will therefore always assume that $q^{-1}(-\infty)$ is $\mathcal{L}$-polar.

Next we derive the *Bernstein-Walsh inequality* for the Green function. It gives an estimate for polynomials in $\mathbb{C}^n$ based on their behavior on a set X. If q is a function on X and P is a polynomial such that $P(z) \leq M \exp(m\, q(z))$ on X, where $m \geq \deg P$, then $\log|\frac{P}{M}|^{1/m} \in \mathcal{L}$ and $\log|\frac{P}{M}|^{1/m} \leq q$. By the definition of $V_{X,q}$ we get the *Bernstein-Walsh inequality*,

$$|P(z)| \leq M \exp(m\, V_{X,q}(z)), \qquad z \in \mathbb{C}^n. \tag{2.5}$$

It is reasonable to ask when our function $V_{X,q}$ is equal to $+\infty$ and furthermore if it is in $\mathcal{L}$. Because the supremum of a family of plurisubharmonic functions is not necessarily upper semicontinuous, therefore as in Proposition 2.1.14 we use the *upper semicontinuous regularization* $V^*_{X,q}$ of $V_{X,q}$.

Proposition 2.3.2 *If X is bounded and E is $\mathcal{L}$-polar then $V^*_{X\cup E,q} = V^*_{X,q}$.*

Proof: Let $w \in \mathcal{L}$, $w|_E = -\infty$ and $w \leq 0$ on X. Then for every $u \in \mathcal{L}$ such that $u \leq q$ on X and for every $k \in \mathbb{N}$

$$\frac{1}{k} w + u \leq V_{X\cup E,q} \leq V^*_{X\cup E,q}.$$

Consequently $u \leq V^*_{X\cup E,q}$ on $\mathbb{C}^n \setminus w^{-1}(-\infty)$, that is almost everywhere in $\mathbb{C}^n$, but because of upper semicontinuity this holds on all $\mathbb{C}^n$, thus implying $V^*_{X,q} \leq V^*_{X\cup E,q}$. The other inequality is obvious since $X \subset X \cup E$. $\square$

The following theorem tells us that either $V_{X,q}$ is of logarithmic growth or identically $+\infty$, and it depends only on X being $\mathcal{L}$-polar or not.

Theorem 2.3.3 *Let q be a function on a set $X \subset \mathbb{C}^n$ such that $q^{-1}(-\infty)$ is not $\mathcal{L}$-polar. Then X is not $\mathcal{L}$-polar if and only if $V^*_{X,q} \in \mathcal{L}$, furthermore $V^*_{X,q} = +\infty$ if X is $\mathcal{L}$-polar.*

Proof: If X is not $\mathcal{L}$-polar then $V^*_{X,q} \in \mathcal{L}$ by Proposition 2.2.4. However, if X is $\mathcal{L}$-polar then by Proposition 2.3.2 $V^*_{X,q} = +\infty$, hence $V^*_{X,q}$ is not in $\mathcal{L}$. $\square$

It is also worth noting when $V_{X,q}$ is not only bounded above by $\log|z|$ but also below.

Proposition 2.3.4 *If X is bounded and q is bounded below, then there is a constant $C \in \mathbb{R}$ such that*

$$\log|z| \le V_{X,q}(z) + C, \qquad z \in \mathbb{C}^n.$$

Proof: Find $r > 0$ such that $X \subset B(r,0)$ and set $M := \inf_X q$. Then $\log\frac{|z|}{r} - M \le q(z)$ on X and the result follows with $C = M + \log(r)$. $\square$

Example 2.3.5 The pluricomplex Green function for the ball. For any complex norm $\|\cdot\|$, let $B = \overline{B}_{\|\cdot\|}(a,r)$ denote its closed ball with centre a and radius r. Then

$$V_B(z) = \log^+\frac{\|z-a\|}{r}, \qquad z \in \mathbb{C}^n,$$

where $\log^+ x = \max\{\log x, 0\}$.

Since all norms on $\mathbb{C}^n$ are equivalent (Lemma 5.14 in [15]), there is a constant C such that $\|\cdot\| \le C|\cdot|$ on $\mathbb{C}^n$. Therefore the function $\log^+\frac{\|z-a\|}{r}$ is in $\mathcal{L}$ and since it is 0 on B, $\log^+\frac{\|z-a\|}{r} \le V_B(z)$. We thus have to show for any $u \in \mathcal{L}$, such that $u \le 0$ on B, that $u(z) \le \log^+\frac{\|z-a\|}{r}$. Note that this clearly holds when $z \in B$.

For such u and $z \in \mathbb{C}^n \setminus B$ we define a function v on $D(0, \|z-a\|/r) \setminus \{0\} \subset \mathbb{C}$ by

$$v(\zeta) = u(a + \zeta^{-1}(z-a)) - \log^+\frac{\|z-a\|}{|\zeta|r}.$$

Then v is subharmonic and $v(\zeta)$ is bounded when $\zeta \to 0$, since $u \in \mathcal{L}$. Therefore v can be extended over 0 to a subharmonic function $\tilde{v}$ on $D(0, \|z-a\|/r)$. Now, $\lim_{|\zeta|\to\|z-a\|/r} v(\zeta) \le 0$, so by the maximum principle $\tilde{v} \le 0$ on $D(0, \|z-a\|/r)$. Observe that $\tilde{v}(1)$ is defined since $\|z-a\|/r \ge 1$ and $v(1) = u(z) - \log^+\frac{\|z-a\|}{r} \le 0$.

Example 2.3.6 From the example above and the *Bernstein-Walsh inequality* we see that for every polynomial $P : \mathbb{C}^n \to \mathbb{C}$, complex norm $\|\cdot\|$ and B as above,

$$|P(z)| \le \|P\|_B \max\left\{\left(\frac{|z-a|}{r}\right)^{\deg P}, 1\right\}.$$

Example 2.3.7 Let $X \subset \mathbb{C}^n$ and q be a function in $\mathcal{L}$. If $u \in \mathcal{L}$ and $u \leq q$ on X then the function $\sup\{q, u\}$ is in $\mathcal{L}$ and $\sup\{q, v\} = q$ on X. Therefore, $V_{X,q} \geq q$ and $V_{X,q} = q$ on X.

2.4 Continuity of the pluricomplex Green functions

Next we prove some basic continuity properties of the pluricomplex Green function. The tools used primarily are properties of plurisubharmonic functions and basic continuity definitions found in Chapter 2.1.

Proposition 2.4.1 *If $K_0 \supset K_1 \supset \cdots$ is a sequence of compact sets in $\mathbb{C}^n$, $K = \cap_j K_j$ and $q : K_0 \to \mathbb{R}$ is lower semicontinuous, then*

$$\lim_{j\to\infty} V_{K_j,q}(z) = V_{K,q}(z), \qquad z \in \mathbb{C}^n.$$

Proof: By definition $V_{K_0,q} \leq V_{K_1,q} \leq \cdots \leq V_{K,q}$, so the limit $\lim_{j\to\infty} V_{K_j,q}$ exists. Let $\varepsilon > 0$ and $u \in \mathcal{L}$, such that $u \leq q$ on K. Define the set $D := \{z \in K_0 : u(z) - \varepsilon < q(z)\}$, then there is j_0 such that D is open and contains K for $j \geq j_0$. This is a consequence of upper semicontinuity of u and lower semicontinuity of q; for fixed $z_0 \in D$ find $\alpha \in \mathbb{R}$, $u(z_0) - \varepsilon < \alpha < q(z_0)$, note that $u(z_0) \neq +\infty$ and $q(z_0) \neq -\infty$ so this α exists. Then we can find a neighbourhood U of z_0 in K_0 such that

$$u(z) - \varepsilon < \alpha < q(z), \qquad z \in U.$$

By compactness of K we can then find $j_0 \in \mathbb{N}$ such that $K_j \subset D$ for $j \geq j_0$. Then $u(z) - \varepsilon \leq V_{K_{j_0},q}(z) \leq \lim_{j\to\infty} V_{K_j,q}(z)$, taking supremum over u and letting $\varepsilon \to 0$ yields $V_{K,q}(z) \leq \lim_{j\to\infty} V_{K_j,q}(z)$, so $V_{K,q} = \lim_{j\to\infty} V_{K_j,q}(z)$. □

Proposition 2.4.2 *For a compact set K in $\mathbb{C}^n$ and a lower semicontinuous function q, the pluricomplex Green function $V_{K,q}$ is lower semicontinuous.*

Proof: According to Proposition 2.2.7 we can approximate any $u \in \mathcal{L}$ from above by a $\mathcal{C}^\infty$ function in $\mathcal{L}$. If $u \leq q$ on K and $\varepsilon > 0$, then for $z_0 \in K$ and $\alpha \in \mathbb{R}$ such that $u(z_0) \leq \alpha \leq q(z_0)$ we can find $u_\lambda \in C^\infty(\mathbb{C}^n) \cap \mathcal{L}$ such that $u_\delta \geq u$ and $u_\delta(z_0) - \varepsilon/2 \leq \alpha$. Since u_δ is continuous and q lower semicontinuous there is a neighbourhood U of z_0 such that $\alpha - \varepsilon/2 \leq q(z)$ for $z \in U$. That is

$$u_\delta(z) - \varepsilon \leq q(z), \qquad z \in U,$$

By compactness of K we can assume this holds on all of K. Therefore, letting $\delta \to 0$ and $\varepsilon \to 0$ we see that $V_{K,q}$ is the supremum of continuous functions and hence lower semicontinuous. □

Proposition 2.4.3 *If $V^*_{X,q} \le q$ on X, then $V_{X,q}$ is upper semicontinuous.*

Proof: Since $V^*_{X,q} \le q < +\infty$ on X, X is not $\mathcal{L}$-polar and thus $V^*_{X,q} \in \mathcal{L}$. Then by definition of $V_{X,q}$, $V^*_{X,q} \le V_{X,q}$. Obviously $V^*_{X,q} \ge V_{X,q}$, so $V_{X,q} = V^*_{X,q}$ and is upper semicontinuous. □

Proposition 2.4.4 *If X is open and q upper semicontinuous then $V^*_{X,q} \le q$ on X and $V_{X,q}$ is therefore upper semicontinuous and in $\mathcal{L}$.*

Proof: Since X is open, we have

$$V^*_{X,q}(z) = \limsup_{w\to z} V_{X,q}(w) \le \limsup_{w \to z} q(w) = q(z), \qquad z \in X.$$

□

Proposition 2.4.5 *If K is compact and $V_{K,q}$ is continuous on K then $V_{K,q}$ is continuous on $\mathbb{C}^n$.*

Proof: We can obviously omit the case when $V_{K,q} = +\infty$, so we assume $V^*_{K,q} \in \mathcal{L}$. By Theorem 2.2.7 there exist $V_\delta \in \mathcal{L} \cap \mathcal{C}^\infty$ such that $V_\delta \searrow V^*_{K,q}$ as $\delta \to 0$. Then $V_\delta \searrow V_{K,q}$ on K, since $V_{K,q}$ is continuous on K, and it is also possible for every $\varepsilon > 0$ to find a $\delta_0 > 0$ such that

$$V_\delta - V_{K,q} < \varepsilon, \quad \text{on } K, \text{ for } 0 < \delta < \delta_0.$$

That is $V_\delta - \varepsilon \le V_{K,q} \le q$ on K and

$$V_\delta - \varepsilon \le V_{K,q} \le V^*_{K,q} \le V_\delta, \qquad \text{on } \mathbb{C}^n,$$

so $V_{K,q}$ is the uniform limit of $\mathcal{C}^\infty$ functions and therefore continuous. □

Proposition 2.4.6 *If $\{X_j\}_{j=0}^\infty$ is an increasing sequence of bounded open sets, $X = \cup_j X_j$ and $q : X \to [-\infty, +\infty[$ upper semicontinuous. Then*

$$\lim_{j\to\infty} V_{X_j,q} = V_{X,q}.$$

Proof: It is clear that $V_{X_0,q} \geq V_{X_1,q} \geq \cdots \geq V_{X,q}$, so the limit exists and $\lim_{j\to\infty} V_{X_j,q} \geq V_{X,q}$. Each function $V_{X_j,q}$ is in $\mathcal{L}$ and $V_{X_j,q} \leq q$ on X_j, hence $\lim_{j\to\infty} V_{X_j,q} \leq q$ on X. Then $\{V_{X_j,q}\}_j$ is a decreasing sequence of plurisubharmonic functions, the limit is therefore plurisubharmonic. Now, $\lim_{j\to\infty} V_{X_j,q} \leq V_{X_0,q} \leq \log^+ |\cdot| + c$ since $V_{X_0,q} \in \mathcal{L}$, therefore is the limit in $\mathcal{L}$ and by definition of $V_{X,q}$, $\lim_{j\to\infty} V_{X_j,q} \leq V_{X,q}$.
□

Proposition 2.4.7 *Assume $X \subset \mathbb{C}^n$ is open and q_j is a decreasing sequence of upper semicontinuous functions on X converging to a function q such that $q^{-1}(-\infty)$ is pluripolar, then*

$$\lim_{j\to\infty} V_{X,q_j} = V_{X,q}.$$

Proof: $V_{X,q_j} \leq q_j$ on X by Proposition 2.4.6, hence $\lim_{j\to\infty} V_{X,q_j} \leq \lim_{j\to\infty} q_j = q$ on X. By monotone convergence, Proposition 2.1.13, we see that $\lim_{j\to\infty} V_{X,q_j} \in \mathcal{L}$ and consequently $\lim_{j\to\infty} V_{X,q_j} \leq V_{X,q}$. But for each j, $V_{X,q_j} \geq V_{X,q}$, so $\lim_{j\to\infty} V_{X,q_j} = V_{X,q}$. □

3

Siciak's approximation theorem

The main subject of this chapter is Siciak's approximation theorem mentioned in Chapter 1. In [21] Siciak gave a proof of this theorem using the extremal function $\Phi_{K,0}$. Siciak [21] originally defined $\Phi_{K,q}$ using extremal points analogous to the definition of Fekete's points for a set in $\mathbb{C}$. Later, the following has become the standard definition of Siciak's extremal function,

$$\Phi_{K,q}(z) = \sup\{|p(z)|^{1/\deg p} : p \text{ a polynomial on } \mathbb{C}^n, |p|^{1/\deg p} \leq \exp(q) \text{ on } K\},$$

for $z \in \mathbb{C}^n$, a compact set $K \subset \mathbb{C}^n$ and q a bounded real function on K.

Here we will prove the approximation theorem using the techniques of Zeriahi [26], introduced for pluricomplex Green functions on parabolic Stein spaces. We use V_K and tools from pluripotential theory along with functional analysis.

First we will however take a closer look at Siciak's extremal function $\Phi_{K,q}$. From the definition of $\Phi_{K,q}$ we can derive a different version of the *Bernstein-Walsh inequality* (2.5). For any polynomial P such that $|P| \leq M \exp(q)^{\deg P}$ on K, we have

$$|P(z)| \leq M\Phi_{K,q}(z)^{\deg P}, \qquad z \in \mathbb{C}^n. \tag{3.1}$$

Our first objective in this chapter is to prove the equality

$$\log \Phi_{K,q} = V_{K,q},$$

when K is compact and q continuous. The properties of the Green function $V_{K,q}$

where studied in the previous chapter, this result will therefore give us information about the continuity of $\Phi_{K,q}$.

3.1 Equivalence of Siciak's and Zahariuta's definitions

Theorem 3.1.1 *Let K be a compact subset of $\mathbb{C}^n$ and $q : K \to \mathbb{R}$ be a continuous function on K. Then $V_{K,q} = \log \Phi_{K,q}$.*

Proof: Clearly $V_{K,q} \geq \log \Phi_{K,q}$, so it is sufficient to prove that if $u \in \mathcal{L}$ and $u \leq q$ on K, then $e^u \leq \Phi_{K,q}$. In order to prove this inequality we take $p \in \mathbb{C}^n$ and $\varepsilon > 0$. By Proposition 2.2.2 the function $v : \mathbb{C}^{n+1} \setminus \{\tilde{z} = (z_0, z) \in \mathbb{C}^{n+1}\,;\, z_0 = 0\}$ defined by $v(z_0, z) = u(z/z_0) + \log|z_0|$ extends to $\tilde{v} \in \mathcal{PSH}(\mathbb{C}^{n+1} \setminus \{0\})$ and then $h = \exp(\tilde{v})$ is in $\mathcal{H}_+^{n+1}$ and defined on $\mathbb{C}^{n+1}$; $h(0) = 0$ by homogeneity. By Theorem 2.1.23 there exists a decreasing sequence $h_j \in \mathcal{H}_+^{n+1} \cap \mathcal{C}(\mathbb{C}^{n+1})$ such that $h_j \searrow h$. By possibly adding a term $\varepsilon_j|\tilde{z}|$ to h_j we may assume that $h_j^{-1}(\{0\}) = \{0\}$. By Theorem 2.2.8(ii) the functions u_j defined by $u_j(z) = \log h_j(1, z)$ are in $\mathcal{L}$, and the sequence $\{u_j\}$ converges to u. Since q is continuous we can choose j_0 such that $u_j \leq q + \varepsilon$ on K for $j \geq j_0$. By Theorem 2.2.8(i) there exists for each $j \geq j_0$ a complex homogeneous polynomial Q_j such that $|Q_j|^{1/d_j} \leq h_j$ and $h_j(1, p) \leq e^{\varepsilon}\,|Q_j(1, p)|^{1/d_j}$, where $d_j = \deg Q_j$. This implies that the polynomial P_j in $\mathbb{C}^n$ defined by $P_j(z) = e^{\varepsilon d_j}\,Q_j(1, z)$ is of degree $\leq d_j$ and $|P_j|^{1/d_j} \leq e^{q+2\varepsilon}$ on K. Hence $|P_j|^{1/d_j} \leq \Phi_{K,q+2\varepsilon}$ and we get

$$e^{u(p)} = \lim_{j\to\infty} e^{u_j(p)} = \lim_{j\to\infty} h_j(1, p) \leq \lim_{j\to\infty} |P_j(p)|^{1/d_j} \leq \Phi_{K,q+2\varepsilon}(p) = e^{2\varepsilon}\Phi_{K,q}.$$

Since $\varepsilon > 0$ is arbitrary we conclude that $e^{u(p)} \leq \Phi_{K,q}(p)$. □

Corollary 3.1.2 *For K a compact subset of $\mathbb{C}^n$ we have*

(i) $V_K = V_{\hat{K}}$ and

(ii) $V_K > 0$ on $\mathbb{C}^n \setminus \hat{K}$.

Proof: (i) Every polynomial P on $\mathbb{C}^n$ such that $|P| \leq 1$ on K satisfies $|P| \leq 1$ on $\hat{K}$ by definition of $\hat{K}$, so $\Phi_K \leq \Phi_{\hat{K}}$ and then $V_K \leq V_{\hat{K}}$. The other inequality is obvious since $K \subset \hat{K}$.

(ii) If $z \in \mathbb{C}^n \setminus \hat{K}$ there is a polynomial P such that $|P(z)| > \|P\|_K$. Set $Q = P/\|P\|_K$, then $|Q| \leq 1$ on K and $\Phi_K(z) \geq |Q(z)|^{1/\deg Q} > 1$, consequently $V_K(z) = \log \Phi_K(z) > 0$. □

3.2 Siciak's approximation theorem

The main motivation for introducing the pluricomplex Green function $V_{X,q}$ was to generalize the Walsh approximation theorem to higher dimensions. This was done by Siciak in [21]. We will now prove this theorem using the method of Zeriahi [26].

The close connection of the Green function with polynomials is best shown in its applications. Siciak studied some of these applications in [21–23], such as polynomial approximation of holomorphic functions, convergence of series of homogeneous polynomials and separately holomorphic functions. Polynomial inequalities are a valuable tool in these studies, specially the *Bernstein-Walsh inequality* which we have derived in two versions, (2.5) and (3.1).

The subject of this section will be polynomial approximation and extension of holomorphic functions. That is, a necessary and sufficient condition for a holomorphic function to have a holomorphic extension to a sublevel set of V_K and the largest such set for an extension to exist.

We begin with some basic definition regarding polynomials and the Green function.

Definition 3.2.1 Set

$$\mathcal{P}_d = \{P \in \mathbb{C}[z_1, \cdots, z_n] : \deg(P) \leq d\}.$$

If f is a holomorphic function in a neighbourhood if a compact set K we let

$$\varepsilon_d(f, K) = \inf_{P \in \mathcal{P}_d} \|f - P\|_K = \inf_{P \in \mathcal{P}_d} \sup_{z \in K} |f(z) - P(z)|.$$

The sublevel sets of V_X will be of great importance, and we introduce a special notation for them,

$$\Omega_R(X) = \{z \in \mathbb{C}^n : V_X(z) < \log R\}.$$

If it is clear from the context what X is, we write Ω_R

Theorem 3.2.2 *Let $K \subset \mathbb{C}^n$ be compact and assume V_K is continuous. Then for all $r > 1$ and $\theta > 0$ there is a constant $c = c(r,\theta) > 0$ such that*

$$\varepsilon_d(f,K) \leq c\,\frac{1}{r^d}\,\|f\|_{\overline{\Omega}_{r+\theta}}, \qquad \text{for all } f \in \mathcal{O}(\overline{\Omega}_{r+\theta}) \text{ and } d \geq 1$$

Proof: For $\alpha \in]0,1[$ we define the non-negative homogeneous function $\tilde{h}_\alpha$ on $\mathbb{C}^{n+1}$ with

$$\tilde{h}_\alpha(z_0,z) = \begin{cases} |z_0|\exp(\alpha\, V_K(\frac{z}{z_0})), & z_0 \in \mathbb{C}\setminus\{0\}, z \in \mathbb{C}^n \\ 0, & z_0 = 0, z \in \mathbb{C}^n. \end{cases}$$

By continuity of V_K and the fact that V_K is in $\mathcal{L}$, the function $\tilde{h}_\alpha$ is continuous and in $\mathcal{H}_+^{n+1}$. We look at K as the compact set $\tilde{K} = \{1\} \times K$ in $\mathbb{C}^{n+1}$ and see that $\tilde{h}_\alpha(1,z) = \exp(\alpha\, V_K(z))$.

Similar to Ω_r we define for every $r \in \mathbb{R}$ the sublevel set for $\tilde{h}_\alpha$,

$$D_r = \{\tilde{z} \in \mathbb{C}^{n+1} : \tilde{h}_\alpha(\tilde{z}) < r\}$$

It is easy to see that $D_{\rho^\alpha} \cap (\{1\} \times \mathbb{C}^n) = \{1\} \times \Omega_\rho$. Note that D_{ρ^α} is a domain of holomorphy since it is a sublevel set of a continuous plurisubharmonic function.

Choose $\alpha \in]0,1[$ such that $r < (r+\theta)^\alpha < r+\theta$ and put $\tilde{K}_r = \{(\lambda, \lambda z) : |\lambda| = r, z \in K\}$. The set $\tilde{K}_r$ is a compact subset of $D_{(r+\theta)^\alpha}$, since $\tilde{h}_\alpha(\lambda,\lambda z) = |\lambda|\exp(\alpha V_K(z)) = r < (r+\theta)^\alpha$.

For $\tilde{f} \in \mathcal{O}(D_{(r+\theta)^\alpha})$, let f denote the corresponding function $f = \tilde{f}(1,\cdot)$ in $\mathcal{O}(\Omega_{r+\theta})$. We now show that the restriction map $T : \mathcal{O}(D_{\rho^\alpha}) \to \mathcal{O}(\Omega_\rho), \tilde{f} \mapsto f$ is both open and surjective.

The set $\{1\} \times \Omega_{r+\theta}$ is a properly imbedded submanifold of $D_{(r+\theta)^\alpha}$ of dimension n. It is locally regularly presented since, i.e., there is a holomorphic functions g such that $\{1\} \times \Omega_{r+\theta} = g^{-1}(0) \cap D_{(r+\theta)^\alpha}$, namely $g(z_0,z) = z_0 - 1$. Then an application of Cartan's Theorem B ([10], Theorem 7.2.8), shows that T is surjective.

The spaces $\mathcal{O}(D_{(r+\theta)^\alpha})$ and $\mathcal{O}(\Omega_{r+\theta})$ are Fréchet spaces and T is continuous, linear and surjective, so the openness of T follows from the *open mapping theorem*, found in [15], Theorem 24.30, Corollary 24.29, and Remark 24.15(c).

The image of $\{\tilde{f} \in \mathcal{O}(D_{(r+\theta)^\alpha}) : \|\tilde{f}\|_{\tilde{K}_r} < 1\}$ under the map T is then an open neighbourhood of 0, i.e., it contains a set of the form $\{f \in \mathcal{O}(\Omega_{r+\theta}) : \|f\|_L < \varepsilon_L\}$, where $L \subset \Omega_{r+\theta}$ is compact and $\varepsilon_L > 0$.

Fix $f \in \mathcal{O}(\overline{\Omega}_{r+\theta})$ and let $\tilde{f} \in \mathcal{O}(D_{(r+\theta)^\alpha})$ be such that $T(\tilde{f}) = f$. Choose $\varepsilon < \varepsilon_L$ and define $\tilde{g} = \varepsilon\tilde{f}/\|f\|_L$. Then $T(\tilde{g}) = \varepsilon\, f/\|f\|_L$, and consequently $\|T(\tilde{g})\|_L = \varepsilon$ and $\|\tilde{g}\|_{\tilde{K}_r} \leq 1$.

Hence, we conclude

$$\|\tilde{f}\|_{\tilde{K}_r} = \|\tilde{g}\|_{\tilde{K}_r}\frac{\|f\|_L}{\varepsilon} \leq \frac{1}{\varepsilon}\|f\|_L, \qquad f \in \mathcal{O}(\overline{\Omega}_{r+\theta}. \tag{3.2}$$

Since $D := D_{(r+\theta)^\alpha}$ is a balanced neighbourhood of the origin in $\mathbb{C}^{n+1}$, we can by Theorem 2.1.21 write,

$$\tilde{f}(\tilde{z}) = \sum_{j=0}^{\infty} \tilde{P}_j(\tilde{z}), \qquad \tilde{z} \in D,$$

where $\tilde{P}_j$ are homogeneous polynomials of degree j and the series is uniformly convergent on compact subsets of D.

This presentation of $\tilde{f}$ gives the following bound on $\varepsilon_d(\tilde{f}, \tilde{K})$,

$$\varepsilon_d(\tilde{f}, \tilde{K}) \leq \sum_{j=d+1}^{\infty} \|P_j\|_{\tilde{K},d}, d \geq 1.$$

As in the proof of Lemma 2.1.20 we apply the Cauchy estimate for every $(1, z) = \tilde{z} \in \tilde{K}$ to the holomorphic function $\lambda \mapsto \tilde{f}(\lambda\,\tilde{z}) = \sum_{j=1}^{\infty} \tilde{P}_j(\tilde{z})\lambda^j$. It yields, with reference to the definition of $\tilde{K}_r$ the estimate

$$|\tilde{P}_j(\tilde{z})| \leq \frac{1}{r^j} \sup_{|\lambda|=r} |\tilde{f}(\lambda\,\tilde{z})| \leq \frac{1}{r^j}\|\tilde{f}\|_{\tilde{K}_r}.$$

This inequality and (3.2) give

$$\varepsilon_d(\tilde{f}, \tilde{K}) \leq \sum_{j=d+1}^{\infty} \|\tilde{P}_j\|_{\tilde{K}} \leq \sum_{j=d+1}^{\infty} \frac{1}{r^j}\,\|\tilde{f}\|_{\tilde{K}_r} = \frac{r}{r-1}\,\frac{\|\tilde{f}\|_{\tilde{K}_r}}{r^{d+1}}.$$

But this also gives us an estimate on $\varepsilon_d(f, K)$, since the set of restrictions of polynomials of $n + 1$ variables and degree $\leq d$ to $\{1\} \times \mathbb{C}^n$ is identical to the set of polynomials of n variables and degree $\leq d$. Thus, $\varepsilon_d(\tilde{f}, \tilde{K}) = \varepsilon_d(f, K)$ and by (3.2),

$$\varepsilon_d(f, K) \leq \frac{1}{\varepsilon(r-1)}\,\frac{\|f\|_L}{r^d}.$$

Setting $c = \frac{1}{\varepsilon}\frac{1}{r-1}$ the result follows, since $L \subset \Omega_{r+\theta}$. $\square$

Finally, there is Siciak's approximation theorem:

Theorem 3.2.3 *Let K be compact subset of $\mathbb{C}^n$ such that V_K is continuous, let $f \in \mathcal{O}(K)$ and $R > 1$. Then f has a holomorphic extension to the sublevel set*

$$\Omega_R = \{z \in \mathbb{C}^n : V_K(z) < \log R\}$$

if and only if

$$\limsup_{d\to\infty} \varepsilon_d(f,K)^{1/d} \leq \frac{1}{R}.$$

Proof: If f is holomorphic in $\Omega_R = \{z \in \mathbb{C}^n : V_K(z) < \log R\}$, then for $\varepsilon \in]0,R[$ set $r = R-\varepsilon$ and $\theta = \varepsilon/2$. Then $f \in \mathcal{O}(\overline{\Omega}_{r+\theta})$ and by Theorem 3.2.2 there is a constant c such that

$$\varepsilon_d(f,K)^{1/d} \leq (c\,\|f\|_{\overline{\Omega}_{r+\theta}})^{1/d}\,\frac{1}{r}.$$

Hence,

$$\limsup_{d\to\infty} \varepsilon_d(f,K)^{1/d} \leq \limsup_{d\to\infty}(c\,\|f\|_{\overline{\Omega}_{R-\varepsilon/2}})^{1/d}\,\frac{1}{r} = \frac{1}{R-\varepsilon}.$$

Since ε was arbitrary the results follows.

To prove the converse statement , let $\varrho > 1/R$. Then the set $\{d \in \mathbb{N} : \varepsilon_d(f,K)^{1/d} > \varrho\}$ is finite, so there is a constant C such that $\varepsilon_d(f,K) \leq C\varrho^d$ for all $d \geq 1$.

For every d there is a polynomial $P_d \in \mathcal{P}_d$ such that $\varepsilon_d(f,K) = \|f - P_d\|_K$. To demonstrate this we only have to note that a polynomial P, deg$P \leq d$ which approximates f better than the zero function, i.e., $\|f-P\|_K \leq \|f\|_K$, satisfies

$$\|P\|_K \leq \|f-P\|_K + \|f\|_K \leq 2\|f\|_K.$$

The coefficients of these polynomials are therefore contained in a compact subset of $\mathbb{C}$ and since $\|f-P\|_K$ is a continuous functions of the coefficients there is a polynomial P_d such that $\|f-P_d\|_K = \varepsilon_d(f,K)$.

Set $Q_1 = P_1$ and $Q_d = P_d - P_{d-1}$ for $d \geq 2$. Then

$$|Q_d(z)| \leq C(\varrho^d + \varrho^{d-1}) = C(1+\frac{1}{\varrho})\varrho^d, \qquad z \in K,\ d \geq 1.$$

By the *Bernstein-Walsh inequality* (3.1), with M as $C(1+\frac{1}{\varrho})\varrho^d$,

$$|Q_d(z)| \leq C(1+\frac{1}{\varrho})\,(\varrho\,\Phi_K(z))^d, \qquad z \in \mathbb{C}^n, d \geq 1.$$

Hence, the series $\sum_{d=1}^{\infty} Q_d$ is uniformly convergent on compact subsets of the sublevel set $\{z \in \mathbb{C}^n : \Phi_K(z) < 1/\varrho\}$ and extending f holomorphically, since this holds for arbitrary $\varrho > 1/R$ it holds on $\{z \in \mathbb{C}^n : V_K(z) \leq \log R\}$. □

Corollary 3.2.4 *Let K be a compact subset of $\mathbb{C}^n$ such that V_K is continuous, and let $f \in \mathcal{O}(K)$. Then, for $R \in]1, +\infty]$,*

$$\Omega_R = \{z \in \mathbb{C}^n : V_K(z) < \log R\}$$

is the largest sublevel set of V_K to which f has a holomorphic extension if and only if

$$\limsup_{d\to\infty} \varepsilon_d(f,K)^{1/d} = \frac{1}{R}.$$

4

Disc formulas for the weighted Green function

This chapter contains a detailed version of our paper, Magnusson and Sigurdsson [14]. We extend the methods of Lárusson and Sigurdsson [13] in order to prove disc formulas for the Green function $V_{X,q}$, when X is an open connected subset of $\mathbb{C}^n$ and q is an upper semicontinuous function on X. Our main result is the following formula

$$V_{X,q}(z) = \inf\Big\{ -\sum_{a\in f^{-1}(H_\infty)} \log|a| + \int_{\mathbb{T}} q\circ f\, d\sigma\,;\, f\in\mathcal{O}(\overline{\mathbb{D}},\mathbb{P}^n), f(\mathbb{T})\subset X, f(0)=z\Big\}. \tag{4.1}$$

Here we let σ be the normalized arc length measure on the unit circle $\mathbb{T}$ and $\mathbb{P}^n$ is the complex projective space viewed in the usual way as the union of the affine space $\mathbb{C}^n$ and the hyperplane at infinity H_∞.

Our approach is the following. Let u be a function in $\mathcal{L}$ such that $u \leq q$ on X. Using the plurisubharmonic extension obtained from Proposition 2.2.2 we derive a formula which defines for us a disc functional J_q. The disc functional J_q will only depend on q and it will satisfy the estimate

$$u(z) \leq J_q(f), \tag{4.2}$$

for any closed analytic disc f in $\mathbb{P}^n$ mapping the origin to z and the unit circle into X. Taking the supremum on the left side of (4.2) over all such u gives us the pluricomplex Green function, and taking the infimum on right side over all discs f gives the envelope of the disc functional J_q. Using Poletsky's theorem we show that the envelope is equal to the Green function, and from that derive formula (4.1).

4.1 Analytic discs and disc functionals

Analytic discs have proved their value in complex analysis, for example the Theorem of Fornæss and Narasihman [1], which determines the plurisubharmonic functions on a complex space using analytic discs, and Poletsky's theorem [20], which gives the largest plurisubharmonic minorant of an upper semicontinuous function.

4.1.1 Notation and basic properties

Definition 4.1.1 An *analytic disc* in a manifold Y is a holomorphic map $f : \mathbb{D} \to Y$ from the unit disc $\mathbb{D}$ in $\mathbb{C}$ into Y. We denote the set of all analytic discs in Y by $\mathcal{O}(\mathbb{D}, Y)$. An analytic disc is said to be *closed* if it can be extended holomorphically to a neighbourhood of $\overline{\mathbb{D}}$. The set of all closed analytic discs in Y will be denoted by $\mathcal{A}_Y$ and the set of all discs in $\mathcal{A}_Y$ which map the unit circle $\mathbb{T}$ into a subset S of Y by $\mathcal{A}_Y^S$.

Definition 4.1.2 A *disc functional* on Y is a map $H : \mathcal{A} \to \overline{\mathbb{R}}$ defined on some subset $\mathcal{A}$ of $\mathcal{O}(\mathbb{D}, Y)$ with values in the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$.

The *envelope* of a disc functional $H : \mathcal{A} \to \overline{\mathbb{R}}$ with respect to the subclass $\mathcal{B}$ of $\mathcal{A}$ yields a function $E_{\mathcal{B}}H$ from Y to $\overline{\mathbb{R}}$ defined by

$$E_{\mathcal{B}}H(x) = \inf\{H(f)\,;\, f \in \mathcal{B}, f(0) = x\}, \qquad x \in Y.$$

As before we view $\mathbb{P}^n$ as the union of the complex space $\mathbb{C}^n$ and the hyperplane at infinity H_∞ (Chapter 2.2) and denote the projection from $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{P}^n$ by π.

Proposition 4.1.3 *If $x \in \mathbb{P}^n$ and $z \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $\pi(z) = x$, then every analytic disc f in $\mathbb{P}^n$ centred at x, i.e., $f(0) = x$, lifts to an analytic disc in $\mathbb{C}^{n+1} \setminus \{0\}$ centred at z.*

Proof: Let $W_m = \{[z_0 : \cdots : z_n] : z_m \neq 0\} \subset \mathbb{P}^n$ and $\psi_m : W_m \to \mathbb{C}^n$ be the local coordinates $[z_0 : \cdots : z_n] \mapsto z_m^{-1}(z_0, \cdots, z_{m-1}, z_{m+1}, \cdots, z_n)$. For $a \in \mathbb{D}$ there exists

$m_a \in \{0, \dots, n\}$ and an open disc $U_a \subset f^{-1}(W_{m_a})$, and we can find a lifting g_a of $f|_{U_a}$ to $\mathbb{C}^{n+1} \setminus \{0\}$ by taking the coordinates $(\gamma_1, \dots, \gamma_n)$ of $\psi_{m_a} \circ f|_{U_a}$ and setting $g_a = (\gamma_1, \dots, \gamma_{m_a - 1}, 1, \gamma_{m_a}, \dots, \gamma_n)$.

We can therefore find a covering $\{U_j\}_{j \in \mathbb{N}}$ and holomorphic functions $g_j : U_j \to \mathbb{C}^{n+1} \setminus \{0\}$ as above such that $\pi(g_j) = f_j|_{U_j}$ for all $j \in \mathbb{N}$. The functions g_j are uniquely defined up to a multiplication with a constant in $\mathbb{C}^*$ and there are holomorphic nonvanishing functions $h_{jk} : U_j \cap U_k \to \mathbb{C}^*$ such that $g_j = h_{jk} g_k$ on $U_j \cap U_k$ for all $j, k \in \mathbb{N}$. Since the functions h_{jk} are uniquely defined, and $g_j = h_{jk} g_k = h_{jk} h_{kl} g_l$, we see that

$$h_{jk}\, h_{kl}\, h_{lj} = 1 \qquad \text{on } U_j \cap U_k \cap U_l, \text{ for all } j, k, l \in \mathbb{N}.$$

Then by Theorem 1.4.5 in [4] there are holomorphic functions $G_j : U_j \to \mathbb{C}^*$ such that $h_{jk} = G_j^{-1} G_k$ on $U_j \cap U_k$ for all $j, k \in \mathbb{N}$. Then $g_j = G_j^{-1} G_k g_k$, that is $G_j g_j = G_k g_k$ on $U_j \cap U_k$. We define a holomorphic function $g : \mathbb{D} \to \mathbb{C}^{n+1} \setminus \{0\}$ by $g(w) = G_j(w) g_j(w)$ when $w \in U_j$. The function g is well defined and satisfies the condition $\pi \circ g = f$. By multiplying g with a suitable constant in $\mathbb{C}^*$, we can assume $g(0) = z$ □

4.1.2 Construction of the disc functional

Take $u \in \mathcal{L}$ and let v denote the extension of $u \circ \pi(z_0, z_1, \cdots, z_n) + \log|z_0|$ to $\mathbb{C}^{n+1} \setminus \{0\}$ obtained from Proposition 2.2.2. Take $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(0) = z \in \mathbb{C}^n$, $f(\mathbb{T}) \subset \mathbb{C}^n$, and let $\tilde{f} = (f_0, \dots, f_n) \in \mathcal{A}_{\mathbb{C}^{n+1} \setminus \{0\}}$ by a lifting of f. By subharmonicity of $v \circ \tilde{f}$ we get

$$u(z) + \log|f_0(0)| = v \circ \tilde{f}(0) \leq \int_{\mathbb{T}} v \circ \tilde{f}\, d\sigma = \int_{\mathbb{T}} u \circ f\, d\sigma + \int_{\mathbb{T}} \log|f_0|\, d\sigma. \tag{4.3}$$

Since $f(\mathbb{T}) \subset \mathbb{C}^n$, the set $f(\mathbb{D})$ has finitely many intersections with H_∞, which means that f_0 has finitely many zeros in $\mathbb{D}$. We write

$$f_0(\zeta) = g_0(\zeta) \prod_{a \in f^{-1}(H_\infty)} \left(\frac{\zeta - a}{1 - \bar{a}\zeta} \right)^{m_{f_0}(a)},$$

where $m_{f_0}(a)$ denotes the multiplicity of a as a zero of f_0 and g_0 is holomorphic and without zeros in some neighbourhood of $\overline{\mathbb{D}}$. Note that $|\frac{\zeta - a}{1 - \bar{a}\zeta}| = 1$ when $\zeta \in \mathbb{T}$. We then have

$$\log|f_0(0)| = \sum_{a\in f^{-1}(H_\infty)} m_{f_0}(a)\log|a| + \log|g_0(0)|, \tag{4.4}$$

and since the product has modulus 1 on $\mathbb{T}$ and $\log|g_0|$ is harmonic in some neighbourhood of $\overline{\mathbb{D}}$, we have

$$\int_{\mathbb{T}} \log|f_0|\, d\sigma = \int_{\mathbb{T}} \log|g_0|\, d\sigma = \log|g_0(0)|. \tag{4.5}$$

By combining (4.4) and (4.5) with (4.3) we derive the inequality

$$u(z) \leq -\sum_{a\in f^{-1}(H_\infty)} m_{f_0}(a)\log|a| + \int_{\mathbb{T}} u\circ f\, d\sigma. \tag{4.6}$$

As in [13] we define the disc functional

$$J : \mathcal{O}(\mathbb{D},\mathbb{P}^n) \to \overline{\mathbb{R}}_+ = [0,+\infty], \qquad J(f) = -\sum_{a\in f^{-1}(H_\infty)} m_{f_0}(a)\log|a|,$$

where we take $J(f) = 0$ if $f^{-1}(H_\infty) = \varnothing$. If $f^{-1}(H_\infty)$ is an infinite set, the sum defining $J(f)$ is taken as the infimum over all finite subsets, which is well defined since the terms are all negative.

Assuming X to be an open subset of $\mathbb{C}^n$ and $q : X \to \mathbb{R}\cup\{-\infty\}$ a Borel measurable function, we can add a mean value term to J and define J_q by

$$J_q : \mathcal{O}(\mathbb{D},\mathbb{P}^n)\cap\mathcal{C}(\overline{\mathbb{D}},\mathbb{P}^n) \to \overline{\mathbb{R}}, \qquad J_q(f) = J(f) + \int_{\mathbb{T}\cap f^{-1}(X)} q\circ f\, d\sigma.$$

In the case when $J(f) = +\infty$ and the integral is $-\infty$ we define $J_q(f)$ as $+\infty$. If $f(\mathbb{T}) \subset X$, then the sum defining $J(f)$ is finite. The special case when we have the constant disc k_z, $\overline{\mathbb{D}} \ni \zeta \mapsto z \in X$, then we have $J(k_z) = 0$, and hence $J_q(k_z) = q(z)$.

The inequality (4.6) implies that for every $u \in \mathcal{L}$ with $u \leq q$ on X and every $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(0) = z$ we have

$$u(z) \leq J_q(f) + \int_{\mathbb{T}\setminus f^{-1}(X)} u\circ f\, d\sigma. \tag{4.7}$$

If $f(\mathbb{T}) \subset X$, then the second term in the right hand side vanishes. If we take the supremum over all $u \in \mathcal{L}$ with $u \leq q$ on X in the left hand side and the infimum over all $f \in \mathcal{B}$ for some subclass $\mathcal{B} \subseteq \mathcal{A}_{\mathbb{P}^n}^X$ in the right hand side, then we arrive at the inequality

$$V_{X,q}(z) \leq E_{\mathcal{A}_{\mathbb{P}^n}^X} J_q(z) \leq E_{\mathcal{B}} J_q(z), \qquad z \in \mathbb{C}^n.$$

What we will prove is the following theorem.

Theorem 4.1.4 *Let* X *be an open connected subset of* $\mathbb{C}^n$ *and* $q : X \to \mathbb{R} \cup \{-\infty\}$ *be an upper semicontinuous function. Then* $V_{X,q} = E_{\mathcal{A}_{\mathbb{P}^n}^X} J_q$*, i.e., for every* $z \in \mathbb{C}^n$ *we have*

$$V_{X,q}(z) = \inf \Big\{ - \sum_{a \in f^{-1}(H_\infty)} m_{f_0}(a) \log |a| + \int_{\mathbb{T}} q \circ f \, d\sigma \, ; \, f \in \mathcal{A}_{\mathbb{P}^n}, f(\mathbb{T}) \subset X, f(0) = z \Big\}.$$

4.1.3 Good sets of analytic discs

We modify the definition from [13] of *good sets* of analytic discs by saying that a subset $\mathcal{B}$ of $\mathcal{A}_{\mathbb{P}^n}$ is *good with respect to the function* q if:

(1) $f(\mathbb{T}) \subset X$ for every $f \in \mathcal{B}$,

(2) for every $z \in \mathbb{C}^n$, there is a disc in $\mathcal{B}$ with centre z,

(3) for every $x \in X$, the constant disc at x is in $\mathcal{B}$, and

(4) the envelope $E_{\mathcal{B}} J_q$ is upper semicontinuous on $\mathbb{C}^n$ and has minimal growth, that is, $E_{\mathcal{B}} J_q - \log^+ |\cdot|$ is bounded above on $\mathbb{C}^n$.

The condition (1) implies that $u(z) \leq J_q(f)$ for every $u \in \mathcal{L}$ with $u \leq q$ and $f \in \mathcal{B}$ with $f(0) = z$, (2) implies that $E_{\mathcal{B}} J_q(z) < +\infty$ for every $z \in \mathbb{C}^n$, (3) implies that $E_{\mathcal{B}} J_q(x) \leq q(x)$ for all $x \in X$, and (4) implies that $V_{X,q}$ is the largest plurisubharmonic function on $\mathbb{C}^n$ dominated by $E_{\mathcal{B}} J_q$, because every plurisubharmonic function less then or equal to $E_{\mathcal{B}} J_q$ is in $\mathcal{L}$ and less then or equal to q on X.

Poletsky's theorem states that for every upper semicontinuous function $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ on a complex manifold Y we have

$$\sup\{u(x) \, ; \, u \in \mathcal{PSH}(Y), u \leq \psi\} = \inf \Big\{ \int_{\mathbb{T}} \psi \circ h \, d\sigma \, ; \, h \in \mathcal{A}_Y, h(0) = x \Big\}, \quad x \in Y.$$

See Poletsky [18], Lárusson and Sigurdsson [11,12], and Rosay [20]. As a consequence we get a disc formula for $V_{X,q}$:

Theorem 4.1.5 *Let X be an open subset of $\mathbb{C}^n$, $q : X \to \overline{\mathbb{R}}$ be a Borel measurable function, and $\mathcal{B}$ be a good class of analytic discs with respect to q. Then*

$$V_{X,q}(z) = \inf \Big\{ \int_{\mathbb{T}} E_{\mathcal{B}} J_q \circ h \, d\sigma \, ; \, h \in \mathcal{A}_{\mathbb{C}^n}, h(0) = z \Big\}, \qquad z \in \mathbb{C}^n.$$

4.1.4 The good set $\mathcal{B}_X$ of analytic discs

The set $\mathcal{A}^X_{\mathbb{P}^n}$ is large and, as seen above in Theorem 4.1.5, we can use any good set of analytic discs instead when evaluating $V_{X,q}$. Therefore we define a convenient set of analytic discs to work with. Let $\mathcal{B}_X$ be the set of all analytic discs in $\mathbb{P}^n$ which are either a constant disc in X or of the following form

$$f_{z,w,r} : \zeta \mapsto w + \frac{|z-w| + r\zeta}{r + |z-w|\zeta} \cdot \frac{r}{|w-z|}(z-w),$$

where $z \in \mathbb{C}^n$, $w \in X \setminus \{z\}$ and $r < \min\{|z-w|, d(w, \partial X)\}$.

Observe that $f_{z,w,r}$ maps $\overline{\mathbb{D}}$ into the projective line through z and w, $\mathbb{T}$ is mapped to a circle with centre w and radius r, 0 is mapped to z, and $-r/|z-w|$ is the only point mapped into H_∞. The conditions on z, w and r ensure that $f_{z,w,r}(\mathbb{T}) \subset X$ and we have the formula

$$J_q(f_{z,w,r}) = \log \frac{|z-w|}{r} + \int_{\mathbb{T}} q \circ f_{z,w,r} \, d\sigma. \tag{4.8}$$

We have to verify that the set $\mathcal{B}_X$ is indeed a good set of analytic discs. It is obvious that $\mathcal{B}_X$ satisfies conditions (1), (2), and (3) in the definition of a good set. By condition (3), $E_{\mathcal{B}_X} J_q(z) \le q(z)$ for all $z \in X$, and since q is upper semicontinuous, this implies that $E_{\mathcal{B}_X} J_q$ is bounded above on every compact subset of X. If we fix $w \in X$ and $r < d(w, \partial X)$, then the integral in (4.8) is bounded above for all $z \in \mathbb{C}^n \setminus B(w,r)$ and it follows that $E_{\mathcal{B}_X} J_q$ is bounded above on every compact subset of $\mathbb{C}^n$ and is of minimal growth. The upper semicontinuity of $E_{\mathcal{B}_X} J_q$ follows from the following lemma:

Lemma 4.1.6 *Assume that $q : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous. For every $z_0 \in \mathbb{C}^n$ and every $\alpha \in \mathbb{R}$ such that $E_{\mathcal{B}_X} J_q(z_0) < \alpha$ there exist $w_0 \in \mathbb{C}^n$, $r_0 > 0$, and a neighbourhood U of z_0 such that $0 < r_0 < \min\{|z - w_0|, d(w_0, \partial X)\}$ and $J_q(f_{z,w_0,r_0}) < \alpha$ for all $z \in U$.*

Proof: Let $f \in \mathcal{B}_X$ such that $f(0) = z_0$ and $J_q(f) < \alpha$. If f is of the form f_{z_0,w_0,r_0} for some $w_0 \in \mathbb{C}^n$ and $0 < r_0 < \min\{d(w_0, \partial X), |z_0 - w_0|\}$, then we can choose a continuous function $\tilde{q} \geq q$ on X such that $J_{\tilde{q}}(f_{z_0,w_0,r_0}) < \alpha$. The continuity of $\tilde{q}$ implies that there exists a neighbourhood U of z_0 such that $r_0 < |z - w_0|$ and $J_{\tilde{q}}(f_{z,w_0,r_0}) < \alpha$ for all $z \in U$. Since $J_q \leq J_{\tilde{q}}$ the statement holds in this case.

Assume now that f is the constant disc at z_0. Then $z_0 \in X$ and $J_q(f) = q(z_0) < \alpha$. Since q is upper semicontinuous, there exists $0 < \delta < d(z_0, \partial X)$ such that $q(z) < \alpha$ for all $z \in B(z_0, \delta)$, the ball with center z_0 and radius δ. Then for every z and w in $B(z_0, \frac{1}{2}\delta)$ and $0 < r < \min\{|z-w|, \frac{1}{2}\delta\}$ we have $\int_{\mathbb{T}} q \circ f_{z,w,r}\, d\sigma < \alpha$. Now choose $w_0 \in B(z_0, \frac{1}{2}\delta)$ and $0 < r_0 < \min\{|z_0 - w_0|, \frac{1}{2}\delta\}$ such that $J_q(f_{z_0,w_0,r_0}) = \log(|z_0 - w_0|/r_0) + \int_{\mathbb{T}} q \circ f_{z_0,w_0,r_0}\, d\sigma < \alpha$. The statement now follows as in the first part of the proof. □

4.2 A disc formula for the weighted Green function

In this section we will prove Theorem 4.1.4, we will use the good class of analytic discs $\mathcal{B}_X$ as well as the following.

For a disc $f \in \mathcal{A}_{\mathbb{P}^n}$ we denote the lifting to $\mathbb{C}^{n+1} \setminus \{0\}$ from Proposition 4.1.3 be $\tilde{f} = (f_0, f_1, \ldots, f_n)$ and we define the function $\varphi : \mathbb{C}^{n+1} \to \mathbb{R} \cup \{-\infty\}$ by $\varphi(\tilde{z}) = \log|z_0|$. Then the Riesz Representation Theorem, (Hörmander [5] Theorem 3.3.6), applied to the subharmonic function $\varphi \circ \tilde{f}$ at the point 0 gives

$$\varphi(\tilde{f}(0)) = \int_{\mathbb{T}} \varphi \circ \tilde{f}\, d\sigma + \frac{1}{2\pi} \int_{\mathbb{D}} \log|\cdot|\Delta(\varphi \circ \tilde{f}).$$

The positive measure $\Delta(\varphi \circ \tilde{f})$ is given by $\Delta(\varphi \circ \tilde{f}) = \sum_{\zeta \in f_0^{-1}(0)} m_{f_0}(\zeta)\delta_\zeta$ which implies

$$\frac{1}{2\pi} \int_{\mathbb{D}} \log|\cdot|\Delta(\varphi \circ \tilde{f}) = \sum_{\zeta \in f_0^{-1}(0)} m_{f_0}(\zeta) \log|\zeta| = -J(\pi \circ \tilde{f})$$

so

$$\int_{\mathbb{T}} \varphi \circ \tilde{f}\, d\sigma = \varphi(\tilde{f}(0)) + J(f).$$

By adding the mean value term of q this implies

$$J_q(f) = \int_{\mathbb{T}} (\varphi \circ \tilde{f} + q \circ f)\, d\sigma - \varphi(\tilde{f}(0)), \qquad f \in \mathcal{A}_{\mathbb{P}^n}^X. \tag{4.9}$$

If X is an open subset of $\mathbb{C}^n$, $q : X \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function and $q_j : X \to \mathbb{R}$ is a decreasing sequence of continuous functions converging to q, then by Proposition 2.4.7, $V_{X,q_j} \searrow V_{X,q}$. It is immediately seen that $J_{q_j}(f) \searrow J_q(f)$ for every $f \in \mathcal{A}^X_{\mathbb{P}^n}$ and as a consequence we get $E_{\mathcal{A}^X_{\mathbb{P}^n}} J_{q_j} \searrow E_{\mathcal{A}^X_{\mathbb{P}^n}} J_q$. This shows that for the proof of Theorem 4.1.4 we may assume that q is continuous.

In the previous section we have seen that $V_{X,q} \leq E_{\mathcal{A}^X_{\mathbb{P}^n}} J_q$ and that $V_{X,q}$ is the largest plurisubharmonic function on $\mathbb{C}^n$ dominated by $E_{\mathcal{B}_X} J_q$. Since $E_{\mathcal{B}_X} J_q$ is upper semicontinuous it can be written as a decreasing sequence of continuous functions. Hence, Theorem 4.1.4 is a direct consequence of Theorem 4.1.5 and the following lemma taking infimum over all continuous $v \geq E_{\mathcal{B}_X}$, $\varepsilon > 0$ and $h \in \mathcal{A}_{\mathbb{C}^n}$ centered at z for a fixed $z \in \mathbb{C}^n$. The lemma is a modification of the Lemma in [13].

Lemma 4.2.1 *Let X be an open connected subset of $\mathbb{C}^n$ and $q : X \to \mathbb{R}$ be continuous. For every $h \in \mathcal{A}_{\mathbb{C}^n}$, every continuous function $v \geq E_{\mathcal{B}_X} J_q$ on $\mathbb{C}^n$, and every $\varepsilon > 0$, there exists a disc $g \in \mathcal{A}^X_{\mathbb{P}^n}$ with $g(0) = h(0)$ and*

$$J_q(g) \leq \int_{\mathbb{T}} v \circ h \, d\sigma + \varepsilon.$$

Proof: Take ζ_0 in $\mathbb{T}$ and set $z_0 = h(\zeta_0)$. Then there is a disc $f \in \mathcal{B}_X$,

$$J_q(f) < E_{\mathcal{B}_X} J_q(z_0) + \varepsilon.$$

This implies that $J_q(f) < v(z_0) + \varepsilon$ and by Lemma 4.1.6 we can assume that f is of the form f_{z_0,w_0,r_0}.

Since the function

$$\zeta \to J_q(f_{h(\zeta),w_0,r_0}) = \log(|h(\zeta) - w_0|/r_0) + \int_{\mathbb{T}} q \circ f_{h(\zeta),w_0,r_0} \, d\sigma$$

is continuous in some neighbourhood of ζ_0 where $r_0 < |h(\zeta) - w_0|$ still holds, there is an open arc U_0 around ζ_0 such that

$$J_q(f_{h(\zeta),w_0,r_0}) < v(h(\zeta)) + \varepsilon/2, \qquad \zeta \in U_0.$$

By compactness, there exists a covering, $U_1, \dots, U_m$, of open arcs on $\mathbb{T}$, points $w_1, \dots, w_m$ and $r_1, \dots, r_m$ such that for every j and $\zeta \in U_j$, the disc $f_{h(\zeta),w_j,r_j}$ is in $\mathcal{B}_X$ and $J_q(f_{h(\zeta),w_j,r_j}) < v(h(\zeta)) + \varepsilon/2$.

Now there are closed arcs $I_j \subset U_j$, $j \in A \subset \{1, \dots, m\}$ which cover $\mathbb{T}$ and have disjoint interiors. By possibly renumbering the arcs and splitting the interval containing 1, we can assume that $A = \{1, \dots, m\}$ and

$$I_j = \{e^{i\theta} : \theta \in [a_j, a_{j+1}]\}, \qquad \text{where } 0 = a_1 < \cdots < a_{m+1} = 2\pi$$

Then

$$\sum_{j=1}^{m} \int_{I_j} J_q(f_{h(\zeta),w_j,r_j})\, d\sigma(\zeta) < \int_{\mathbb{T}} v \circ h\, d\sigma + \varepsilon/2 \tag{4.10}$$

Since X is connected, w_j and w_{j+1} can be joined by a $\mathcal{C}^\infty$ path $\alpha_j : [0,1] \to X$ where $\alpha_j(0) = w_j$ and $\alpha_j(1) = w_{j+1}$; assuming $w_{m+1} = w_1$, we can also find $\mathcal{C}^\infty$ functions $\beta_j : [0,1] \to (0,\infty)$, such that $\beta_j(0) = r_j$, $\beta_j(1) = r_{j+1}$ and $\beta_j(t) < \min\{d(\alpha_j(t), \partial X), |z_0 - \alpha_j|\}$, for all $t \in [0,1]$. We may assume that derivatives of all orders of α_j and β_j vanish at 0 and 1.

Now choose

$$C > \sum_{j=1}^{m} \sup_{\zeta \in I_j, t \in [0,1]} |J_q(f_{h(\zeta),w_j,r_j}) - J_q(f_{h(\zeta),\alpha_j(t),\beta_j(t)})| \tag{4.11}$$

and $\delta > 0$ such that $C\delta < \varepsilon/2$ and $\delta < \min_j(a_{j+1} - a_j)$.

Split each I_j into the subarcs $K_j = \{e^{i\theta} : \theta \in [a_j, a_{j+1} - \delta]\}$ and $L_j = \{e^{i\theta} : \theta \in [a_{j+1} - \delta, a_{j+1}]\}$, and define the $\mathcal{C}^\infty$ loop $\gamma : \mathbb{T} \to X$ by

$$\gamma(\zeta) = \begin{cases} w_j, & \zeta \in K_j, j = 1, \dots, m, \\ \alpha_j((\theta - a_{j+1} + \delta)/\delta), & \zeta = e^{i\theta} \in L_j, j = 1, \dots, m, \end{cases}$$

the $\mathcal{C}^\infty$ function $\rho : \mathbb{T} \to (0,\infty)$ by

$$\rho(\zeta) = \begin{cases} r_j, & \zeta \in K_j, j = 1, \dots, m, \\ \beta_j((\theta - a_{j+1} + \delta)/\delta), & \zeta = e^{i\theta} \in L_j, j = 1, \dots, m, \end{cases}$$

and finally the $\mathcal{C}^\infty$ family of analytic discs in $\mathcal{A}^X_{\mathbb{P}^n}$,

$$F(\cdot, \zeta) = f_{h(\zeta),\gamma(\zeta),\rho(\zeta)}, \qquad \zeta \in \mathbb{T}.$$

By (4.10) and (4.11)

$$\int_{\mathbb{T}} J_q(F(\cdot,\zeta))\,d\sigma(\zeta) = \sum_{j=1}^m \int_{K_j} J_q(f_{h(\zeta),w_j,r_j})\,d\sigma(\zeta) + \int_{L_j} J_q(f_{h(\zeta),\gamma(\zeta),\rho(\zeta)})\,d\sigma(\zeta)$$

$$= \sum_{j=1}^m \int_{I_j} J_q(f_{h(\zeta),w_j,r_j})\,d\sigma(\zeta) + \int_{L_j} \big(J_q(f_{h(\zeta),\gamma(\zeta),\rho(\zeta)}) - J_q(f_{h(\zeta),w_j,r_j})\big)\,d\sigma(\zeta)$$

$$< \sum_{j=1}^m \int_{I_j} J_q(f_{h(\zeta),w_j,r_j})\,d\sigma(\zeta) + C\delta < \int_{\mathbb{T}} v\circ h\,d\sigma + \varepsilon \tag{4.12}$$

Let $\tilde{h} = (1,h)$ be a lifting of h to $\mathcal{A}^X_{\mathbb{C}^{n+1}\setminus\{0\}}$ and $\tilde{f}_{z,w,r}$ the lifting of $f_{z,w,r}$ given by

$$\tilde{f}_{z,w,r}(\xi) = \left(1+\frac{|z-w|\xi}{r}, \left(\frac{|z-w|\xi}{r}+1\right)w + \left(\frac{r\xi}{|z-w|}+1\right)(z-w)\right).$$

Then the lifting $\tilde{F}(\cdot,\zeta) = \tilde{f}_{h(\zeta),\gamma(\zeta),\rho(\zeta)}$ of F satisfies $\tilde{F}(0,\cdot) = \tilde{h}$ on $\mathbb{T}$.

Take $r>1$ such that $h\in\mathcal{O}(D_r,\mathbb{C}^n)$ and $F(\cdot,\zeta)\in\mathcal{O}(D_r,\mathbb{P}^n)$ for all $\zeta\in\mathbb{T}$ where $D_r=\{z\in\mathbb{C} : |z|<r\}$, and define $\tilde{F}_j\in\mathcal{O}(D_r\times(D_r\setminus\{0\}),\mathbb{C}^{n+1})$, $j\geq 1$, by

$$\tilde{F}_j(\xi,\zeta) = \tilde{h}(\zeta) + \sum_{k=-j}^{j}\left(\frac{1}{2\pi}\int_0^{2\pi}\big(\tilde{F}(\xi,e^{i\theta}) - \tilde{h}(e^{i\theta})\big)e^{-ik\theta}d\theta\right)\zeta^k$$

Since the function $\theta\mapsto\tilde{F}(\xi,e^{i\theta})-\tilde{h}(e^{i\theta})$ is $\mathcal{C}^\infty$ with period 2π, its Fourier series converges uniformly on $\mathbb{R}$ to the function itself. Hence, the sequence $\tilde{F}_j$ converges uniformly on $\{\xi\}\times\mathbb{T}$ for each $\xi\in D_r$. The convergence is uniform on $D_t\times\mathbb{T}$ for $t\in(1,r)$. So fix $t\in(1,r)$, then by uniform convergence of $\tilde{F}_j$ and the fact that $\tilde{F}(D_r\times\mathbb{T})\subset\mathbb{C}^{n+1}\setminus\{0\}$ and $F(\mathbb{T}\times\mathbb{T})\subset X$ we can find j large enough such that $\tilde{F}_j(D_t\times\mathbb{T})\subset\mathbb{C}^{n+1}\setminus\{0\}$ and $\tilde{F}_j(\mathbb{T}\times\mathbb{T})\subset\pi^{-1}(X)$. For such j define $F_j=\pi\circ\tilde{F}_j : D_t\times\mathbb{T}\to\mathbb{P}^n$.

The 0th coordinate of $\tilde{F}$ is $\tilde{F}_0(\xi,\cdot) = |h-\gamma|\xi/\rho+1$ and $\tilde{h}_0=1$ so $\tilde{F}_{j0}(\zeta,\cdot) = \chi_j\xi+1$, where $\theta\mapsto\chi_j(e^{i\theta})$ is the jth partial sum of the Fourier series of $\theta\mapsto|h(e^{i\theta})-\gamma(e^{i\theta})|/\rho(e^{i\theta})$, that is the 0th coordinate of the Fourier series for $\tilde{F}(\xi,\cdot)-\tilde{h}$. So the disc $\xi\mapsto F_j(\xi,\zeta)$ sends only $\xi=-1/\chi_j(\zeta)$ to H_∞.

Hence,

$$\begin{aligned} J_q(F_j(\cdot,\zeta)) &= \log|\chi_j(\zeta)| + \int_{\mathbb{T}} q\circ F_j(\xi,\zeta)d\sigma(\xi) \\ &\to \log|h(\zeta)-\gamma(\zeta)|/\rho(\zeta) + \int_{\mathbb{T}} q\circ F(\xi,\zeta)d\sigma(\xi) = J_q(F(\cdot,\zeta)) \end{aligned}$$

uniformly for $\zeta\in\mathbb{T}$. Thus, by (4.12), we can fix j large enough so

$$\int_{\mathbb{T}} J_q(F_j(\cdot,\zeta))d\sigma(\zeta) < \int_{\mathbb{T}} v\circ h d\sigma + \varepsilon.$$

For every $\xi\in D_r$ the map $\zeta\mapsto \tilde F_j(\xi,\zeta)-\tilde h(\zeta)$ has a pole of order at most j at the origin, and for every $\zeta\in D_r\setminus\{0\}$, the map $\xi\mapsto\tilde F_j(\xi,\zeta)-\tilde h(\zeta)$ has a zero at the origin. Hence, $(\xi,\zeta)\mapsto\tilde F_j(\xi\zeta^k,\zeta)$ extends to a holomorphic map $\overline{\mathbb{D}}\times\overline{\mathbb{D}}\to\mathbb{C}^{n+1}$ for every $k\geq j$.

Since $\tilde F_j(0,\zeta)=\tilde h(\zeta)\in\mathbb{C}^{n+1}\setminus\{0\}$ for all $\zeta\in D_r\setminus\{0\}$, there is $\delta>0$ such that $\tilde F_j(\xi\zeta^k,\zeta)\in\mathbb{C}^{n+1}\setminus\{0\}$ for all $k\geq j$ and $(\xi,\zeta)\in D_\delta\times\overline{\mathbb{D}}$. Since $\tilde F_j(\xi,\zeta)\in\mathbb{C}^{n+1}\setminus\{0\}$ for all $(\xi,\zeta)\in\overline{\mathbb{D}}\times\mathbb{T}$, there is $\tau<1$ such that $\tilde F_j(\xi,\zeta)\in\mathbb{C}^{n+1}\setminus\{0\}$ for all $(\xi,\zeta)\in\overline{\mathbb{D}}\times(\overline{\mathbb{D}}\setminus D_\tau)$ and all $k\geq j$. Choose $k\geq j$ large enough that $|\xi\zeta^k|<\delta$ for all $(\xi,\zeta)\in\overline{\mathbb{D}}\times D_\tau$. Then there is an $s\in(1,t)$ such that $\tilde F_j(\xi\zeta^k,\zeta)\in\mathbb{C}^{n+1}\setminus\{0\}$ for all $(\xi,\zeta)\in D_s\times D_s$.

Now define $\tilde G\in\mathcal{O}(D_s\times D_s,\mathbb{C}^{n+1}\setminus\{0\})$ by $\tilde G(\xi,\zeta)=\tilde F_j(\xi\zeta^k,\zeta)$ and let $G=\pi\circ\tilde G$. Remember that $\varphi:\mathbb{C}^{n+1}\to\mathbb{R}\cup\{-\infty\}$ is defined as $\varphi(z_0,z_1,\ldots,z_n)=\log|z_0|$. Using formula (4.9) on the analytic disc $\xi\mapsto\tilde G(\xi,\cdot)$ we see that

$$J_q(G(\xi,\cdot)) = \int_{\mathbb{T}}\left(\varphi\circ\tilde G(\xi,\cdot)+q\circ G(\xi,\cdot)\right)\,d\sigma(\xi)-\varphi(\tilde G(0,\cdot)),$$

but $\tilde{G}(0,\cdot) = \tilde{h} = (1,h)$, so $\varphi(\tilde{G}(0,\cdot)) = 0$. Therefore,

$$\begin{aligned}
\int_{\mathbb{T}} J_q(G(\cdot,\zeta)\ d\sigma(\zeta) &= \int_{\mathbb{T}^2} \big(\varphi\circ\tilde{G} + q\circ G\big)\ d(\sigma\times\sigma) \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \big(\varphi(\tilde{F}_j(e^{i(t+k\theta)}, e^{i\theta})) + q(\pi(\tilde{F}_j(e^{i(t+k\theta)}, e^{i\theta})))\big)\ dtd\theta \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \big(\varphi(\tilde{F}_j(e^{it}, e^{i\theta})) + q(\pi(\tilde{F}_j(e^{it}, e^{i\theta})))\big)\ dtd\theta \\
&= \int_{\mathbb{T}^2} \big(\varphi\circ\tilde{F}_j + q\circ\pi\circ\tilde{F}_j\big)\ d(\sigma\times\sigma) = \int_{\mathbb{T}} J_q(F_j(\cdot,\zeta))\ d\sigma(\zeta) \\
&< \int_{\mathbb{T}} v\circ h\ d\sigma + \varepsilon.
\end{aligned}$$

By the mean value theorem there is a $\theta_0 \in [0, 2\pi]$ such that

$$\begin{aligned}
\int_{\mathbb{T}} J_q(G(\cdot,\zeta)\ d\sigma(\zeta) &= \int_{\mathbb{T}^2} \Big(\varphi\circ\tilde{G} + q\circ\pi\circ\tilde{G}\Big)\ d(\sigma\times\sigma) \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \Big(\varphi(\tilde{G}(e^{i(t+\theta)}, e^{it})) + q(G(e^{i(t+\theta)}, e^{it}))\Big)\ dt\, d\theta \\
&\geq \frac{1}{2\pi}\int_0^{2\pi} \Big(\varphi(\tilde{G}(e^{i(t+\theta_0)}, e^{it})) + q(G(e^{i(t+\theta_0)}, e^{it}))\Big)\ dt
\end{aligned}$$

Now define $\tilde{g}(\zeta) = \tilde{G}(e^{i\theta_0}\zeta, \zeta)$ for $\zeta \in D_s$ and $g = \pi\circ\tilde{g}$. Then $g(0) = \pi(\tilde{g}(0)) = \pi(\tilde{G}(0,0)) = \pi(1, h(0)) = h(0)$ and $g(\mathbb{T}) \subset \pi(\tilde{G}(\mathbb{T}\times\mathbb{T})) \subset X$. So $g \in \mathcal{A}_{\mathbb{P}^n}^X$ and

$$\begin{aligned}
J_q(g) &= \int_{\mathbb{T}} \varphi\circ\tilde{g}\ d\sigma + \int_{\mathbb{T}} q\circ g\ d\sigma \\
&= \frac{1}{2\pi}\int_0^{2\pi} \big(\varphi\circ\tilde{G}(e^{i(\theta_0+t)}, e^{it}) + q\circ G(e^{i(\theta_0+t)}, e^{it})\big)\ d\sigma \\
&\leq \int_{\mathbb{T}} J_q(G(\cdot,\zeta)d\sigma(\zeta) < \int_{\mathbb{T}} v\circ h\ d\sigma + \varepsilon.
\end{aligned}$$

$\square$

4.2.1 Conclusion

We have shown that when $X \subset \mathbb{C}^n$ is a domain and $q : X \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function, then

$$V_{X,q}(z) = \inf\Big\{ - \sum_{a \in f^{-1}(H_\infty)} m_{f_0}(a) \log |a| + \int_{\mathbb{T}} q \circ f \, d\sigma \, ; \, f \in \mathcal{A}_{\mathbb{P}^n}, f(\mathbb{T}) \subset X, f(0) = z \Big\}.$$

Note that this is not the same formula as (4.1), since here the multiplicity of the intersection of f with H_∞ is included in the sum. In order to omit the multiplicity and show that formula (4.1) follows from Theorem 4.1.4 we need the following proposition from [13].

Proposition 4.2.2 *Let $X \subset \mathbb{C}^n$ be open and $f \in \mathcal{A}_{\mathbb{P}^n}$, $f(0) \notin H_\infty$. Then there is a disc $g \in \mathcal{A}_{\mathbb{P}^n}$ with $g(0) = f(0)$, $m_{g_0} = 1$ on $g^{-1}(H_\infty)$ and $J(f) = J(g)$, such that g is uniformly as close to f on $\overline{\mathbb{D}}$ as we wish, in particular $g(\mathbb{T}) \subset X$ if $f(\mathbb{T}) \subset X$.*

Proof: Since $f(\mathbb{T}) \subset \mathbb{C}^n$ f intersects H^∞ in finitely many points $a_1, \dots, a_j \in \mathbb{D} \setminus \{0\}$ with multiplicities $m_j = m_{f_0}(a_j)$. Let $\tilde{f} \in A_{\mathbb{C}^{n+1}\setminus\{0\}}$ be a lifting of f. By Lemma 3.1 in [12], with α as the characteristic function of $\{(0,z) : z \in \mathbb{C}^n \setminus \{0\}\}$ in $\mathbb{C}^{n+1} \setminus \{0\}$, we obtain $\tilde{g} \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{0\}}$ arbitrarily uniformly close to $\tilde{f}$ on $\overline{\mathbb{D}}$ such that $\tilde{f}(0) = \tilde{g}(0)$, the zeros $c_1, \dots, c_m$ of $\tilde{g}_0$ in $\mathbb{D}$ all have multiplicity 1, $m = m_1 + \cdots + m_k$ and

$$\sum_{j=1}^{m} \log |c_j| = \sum_{j=1}^{k} m_j \log |a_j|.$$

Finally take $g = \pi \circ \tilde{g}$. □

Now observe that the upper semicontinuity of q implies that for every $\varepsilon > 0$ and every $f \in \mathcal{A}_{\mathbb{P}^n}^X$ there exists a continuous function $\tilde{q} \geq q$ on X such that $\int_{\mathbb{T}} \tilde{q} \circ f \, d\sigma < \int_{\mathbb{T}} q \circ f \, d\sigma + \varepsilon$. By Proposition 4.2.2, every $f \in \mathcal{A}_{\mathbb{P}^n}$ can be approximated uniformly on $\overline{\mathbb{D}}$ by $g \in \mathcal{A}_{\mathbb{P}^n}$, such that all the zeros of g_0 are simple, $g(0) = f(0)$, and $J(g) = J(f)$. Since $\tilde{q}$ is continuous we can choose g such that $\int_{\mathbb{T}} \tilde{q} \circ g \, d\sigma < \int_{\mathbb{T}} \tilde{q} \circ f \, d\sigma + \varepsilon$. This gives that $J_q(g) \leq J_{\tilde{q}}(g) < J_q(f) + 2\varepsilon$ and we conclude that the infimum in formula (4.1) and Theorem 4.1.4 are equal.

The result is, for a domain $X \subset \mathbb{C}^n$ and an upper semicontinuous function $q : X \to \mathbb{R} \cup \{-\infty\}$,

$$V_{X,q}(z) = \inf\Big\{ -\sum_{a \in f^{-1}(H_\infty)} \log|a| + \int_{\mathbb{T}} q \circ f \, d\sigma \,;\, f \in \mathcal{O}(\overline{\mathbb{D}}, \mathbb{P}^n), f(\mathbb{T}) \subset X, f(0) = z \Big\}.$$

4.3 Applications of the disc formula

The purpose of this section is to give some examples of the applications of the disc formula for the weighted pluricomplex Green function. First, we calculate the pluricomplex Green function for the unit ball when q is of the form $k\log|\cdot|$, $k \in [0,1]$. Secondly, we derive an equality between the Green function for a closed ball and for an open ball, and then use that to give some result regarding the continuity of the Green function for compact sets.

By looking better at the inequality

$$V_{X,q}(z) \le E_{\mathcal{A}^X_{\mathbb{P}^n}} J_q(z) \le E_{\mathcal{B}} J_q(z), \qquad z \in \mathbb{C}^n.$$

we see that, if we can find using $\mathcal{B}$ and J_q a function $u \in \mathcal{L}$ such that $E_{\mathcal{B}} J_q \le u$ and $u \le q$ on X then $u = V_{X,q}(z)$. Here, $\mathcal{B}$ can be $\mathcal{A}^X_{\mathbb{P}^n}$, $\mathcal{B}_X$ or any other good set of analytic discs with respect to q.

Example 4.3.1 Here q will be $k\log|\cdot|$, where $0 \le k \le 1$ and the set will be the unit ball $\mathbb{B}$. We will show that $V_{\mathbb{B},k\log|z|}(z) = \max\{\log|z|, k\log|z|\}$.

To do that we use the family $\mathcal{B}_{\mathbb{B}}$ from Section 4.1.4, consisting of the constant discs in $\mathbb{B}$ and all the discs

$$f_{z,w,r}(\zeta) = w + \frac{|z-w| + r\zeta}{r + |z-w|\zeta} \frac{r}{|z-w|}(z-w)$$

where $z \in \mathbb{C}^n$, $w \in \mathbb{B} \setminus \{z\}$ and $r < \min\{|z-w|, d(w, \partial\mathbb{B})\}$.

First we fix $w = 0$, then

$$J_{k\log|\cdot|}(f_{z,0,r}) \;=\; \log\frac{|z|}{r} + \int_{\mathbb{T}} k\log\Big|r\frac{|z| + r\zeta}{r + |z|\zeta}\frac{z}{|z|}\Big| \, d\sigma(\zeta) = \log\frac{|z|}{r} + k\log r$$

and since $k \le 1$ this is an decreasing function of r.

If $z \in \mathbb{B}$ then this implies that

$$V_{\mathbb{B},k\log|\cdot|}(z) \leq \inf\Big\{\log\frac{|z|}{r} + k\log r : r < |z|\Big\} = k\log|z|. \tag{4.13}$$

However, when $z \in \mathbb{C}^n \setminus \mathbb{B}$ then $|z-w| > d(w, \partial\mathbb{B})$ and

$$V_{\mathbb{B},k\log|\cdot|}(z) \leq \inf\Big\{\log\frac{|z|}{r} + k\log r : r < 1\Big\} = \log|z|. \tag{4.14}$$

The function

$$u(z) := \begin{cases} k\log|z| & \text{on } \mathbb{B} \\ \log|z| & \text{on } \mathbb{C}^n \setminus \mathbb{B} \end{cases}$$

is then continuous and the condition $0 \leq k \leq 1$ ensures that it is plurisubharmonic. It is then obvious that u is in $\mathcal{L}$ and that $u \leq V_{\mathbb{B},k\log|\cdot|}$.

Hence, $u \in \mathcal{L}$, $u \leq q$ on $\mathbb{B}$ and $V_{\mathbb{B},k\log|\cdot|} \leq E_{\mathcal{B}}J_q(z) \leq u \leq V_{\mathbb{B},k\log|\cdot|}$ so $u = V_{\mathbb{B},k\log|\cdot|}$.

When $k = 0$ in the example above we get a equality of the Green function for the closed unit ball (Example 2.3.5) and the open unit ball in the unweighted case, $q = 0$. It interesting to see if this is the genereal case, that is for an arbitrary function q.

Proposition 4.3.2 *If q is upper semicontinuous real function on $\overline{B}(a,s)$, where $a \in \mathbb{C}^n$ and $s > 0$, then $V_{B(a,s),q} = V_{\overline{B}(a,s),q}$.*

Proof: We can assume $a = 0$ and we let B denote $B(0,s)$. It is obvious that $V_{\overline{B},q} \leq V_{B,q}$. Since B is open, $V_{B,q}$ is in $\mathcal{L}$ and $V_{B,q} \leq q$ on B by Proposition 2.4.4. To show that $V_{B,q} \leq V_{\overline{B},q}$ it is therefore enough to show that $V_{B,q} \leq q$ on ∂B.

Let $z \in \partial B$ and $\varepsilon > 0$. Since q is upper semicontinuous there is a neighbourhood U of z such that $q(\zeta) \leq q(z) + \varepsilon/2$ for $\zeta \in U$. Then there is a $\lambda \in]0, s[$ such that

$$B\Big(\lambda\frac{z}{|z|}, s-\lambda\Big) \subset U.$$

Let $w = \lambda z/|z|$ and $r > 0$ such that $(s-\lambda)/e^{\varepsilon/2} < r < s - \lambda$. Then

$$\log\frac{s-\lambda}{r} \leq \varepsilon/2, \quad \text{that is} \quad \log\frac{|z-w|}{r} < \varepsilon/2.$$

Consequently, $f_{z,w,r}(\mathbb{T}) \subset B(\lambda z/|z|, r) \subset B \cap U$, where $f_{z,w,r}$ is the disc from the class $\mathcal{B}_B$ in Section 4.1.4, and since the disc $f_{z,w,r}$ is in $\mathcal{A}^B_{\mathbb{P}^n}$ we get the following

$$V_{B,q}(z) \leq J_q(f_{z,w,r}) = \log\frac{|z-w|}{r} + \int_{\mathbb{T}} q \circ f_{z,w,r} d\sigma < q(z) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $V_{B,q}(z) \leq q(z)$. □

From the proposition above we can prove a weighted version of Corollary 5.1.5 in [8], which gives a sufficient condition on the set K and the function q such that $V_{K,q}$ is continuous.

Corollary 4.3.3 *Let $K \subset \mathbb{C}^n$ be compact set and let q be a continuous real function on a neighbourhood of K. Then, for $\varepsilon > 0$, the Green function $V_{K_\varepsilon,q}$ is continuous, where $K_\varepsilon = \{z \in \mathbb{C}^n : d(z, K) \leq \varepsilon\}$. Moreover,*

$$\lim_{\varepsilon\to 0} V_{K_\varepsilon,q} = V_{K,q}.$$

Proof: The functions $V_{K_\varepsilon,q}$ and $V_{K,q}$ are lower semicontinuous be Proposition 2.4.2 and the limit is valid by Proposition 2.4.1, therefore we only have to show that $V_{K_\varepsilon,q}$ is upper semicontinuous.

When q is continuous it follows from Propositions 2.4.2 and 2.4.4 that the Green function for the closed ball is lower semicontinuous and the Green function for the open ball is upper semicontinuous, since these function coincide by Proposition 4.3.2, it is continuous. Note that we can write K_ε as $\cup_{a\in K}\overline{B}(a, \varepsilon)$. Let $a \in K$, then $V_{B(a,\varepsilon),q} \leq q$ on $B(a, \varepsilon)$. The Green function and the function q are both continuous, therefore $V_{B(a,\varepsilon),q} \leq q$ on $\overline{B}(a, \varepsilon)$, and since $V_{K_\varepsilon,q} \leq V_{B(a,\varepsilon),q}$ we get

$$V^*_{K_\varepsilon,q} \leq V_{B(a,\varepsilon),q} \leq q.$$

The function $V_{K_\varepsilon,q}$ is then upper semicontinuous by Proposition 2.4.3. □

A

Index of symbols

A^c	compliment of A
∂A	boundary of A
$f\|_A$	restriction of f to A
$d(x, A)$	$\inf\{\|z-a\| : a \in A\}$
$B(a, r)$	open ball with center a and radius r
$D(a, r)$	$B(a, r)$ in $\mathbb{C}$
$\mathbb{B}$	unit ball in $\mathbb{C}^n$
$\mathbb{D}$	unit disc in $\mathbb{C}$
$\mathbb{T}$	$\partial\mathbb{D}$
$\mathbb{Z}_+$	$n \in \mathbb{Z}$, $n \geq 0$
$\mathbb{N}$	$n \in \mathbb{Z}$, $n > 0$
$\log^+ x$	$\max\{\log x, 0\}$
$\mathcal{PSH}(A)$	plurisubharmonic functions on A
$\mathcal{O}(A, B)$	holomorphic functions from A to B
$\mathcal{O}(A)$	$\mathcal{O}(A, \mathbb{C})$
$\mathcal{A}_A$	$\mathcal{O}(\overline{\mathbb{D}}, A)$
$\mathcal{A}_A^B$	$f \in \mathcal{A}_A$, $f(\mathbb{T}) \subset B$
$\mathcal{L}$	$f \in \mathcal{PSH}(\mathbb{C}^n)$, $f(z) \leq \log^+ \|z\| + C$, C a constant
$\|z\|$	Euclidean norm of $z \in \mathbb{C}^n$, $\|z\| = (z_1\overline{z}_1 + \ldots + z_n\overline{z}_n)^{1/2}$
$\\|f\\|_K$	$\sup_{z\in K} \|f(z)\|$, where f is a function on a compect set K

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