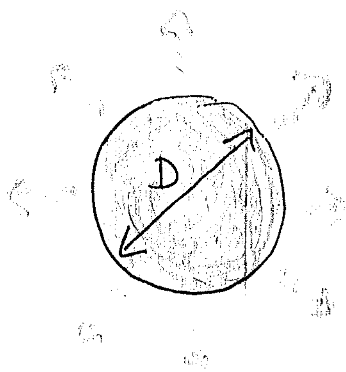
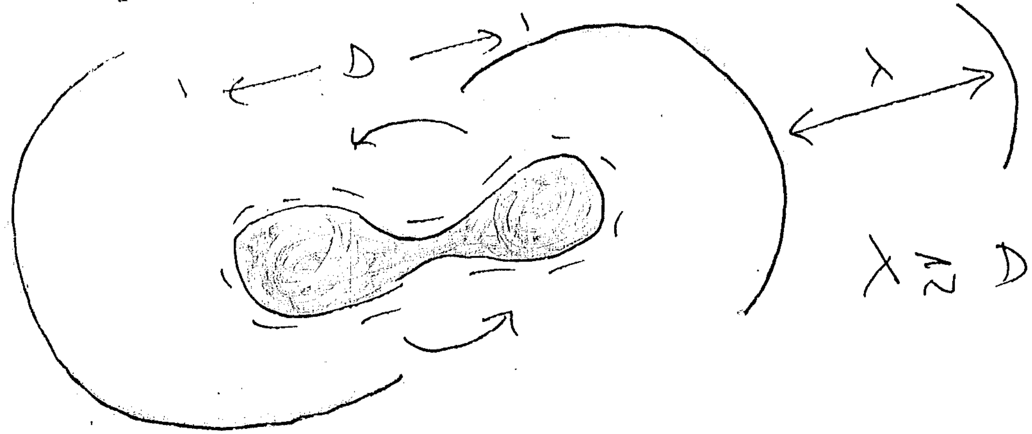
RAFSEGULBYLGJUR	ÞYNGDARBYLGJUR
 $\lambda <<< D$	 $\lambda \gtrsim D$
Misfasa geislun frá rafeindum, atómum og/eða sameindum	Samfasa geislun frá bjögun stjarnfræðilegra fyrirbæra

	10^{-15} m	10^{-7} m	100 m	λ	10 km	10^5 km	10^9 km	0.1 pc	10 pc	10 Mpc	10 Gpc
	10^{24} Hz	10^{16} Hz	10^7 Hz	f	10^4 Hz	1 Hz	10^{-4} Hz	10^{-7} Hz	10^{-9} Hz	10^{-15} Hz	10^{-18} Hz

PSR 1913+16

γ-geislun	Sýnilegt ljós	Útvarpsgeislun	
Sívalningar	LIGO	LISA	ms-tifstjörnur	CMB

Geimgeislar, Hamfarir, Heitt gas — Stjörnur — Gas / Ryk, Samhraðlageislun

Mikil víxlverkun við efni

Svarthol, Sprengistjö., Tifstjörnur, Árekstrar — Risasvarthol, Tvístirni — Frumheimur, Geimstrengir — Frumheimur

Lítil sem engin víxlverkun við efni

ÞYNGDARFRÆÐI EINSTEINS

Sviðsjöfnur

$$\underbrace{R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}}_{G^{\alpha\beta}} + \underbrace{\Lambda g^{\alpha\beta}}_{\text{má færa}} = \overbrace{\frac{8\pi G}{c^4}}^{\varkappa} T^{\alpha\beta}$$

10 óháðar og ólínulegar breiðgerar 2. stigs hlutafleiðujöfnur á Lorentz-viðáttu fyrir 10 óháð föll (mætinaþáttuna) $g^{\alpha\beta}$ (þinametsin)

Bianchi jöfnur

$$\boxed{G^{\alpha\beta}{}_{;\beta} = 0} \quad \Longrightarrow \quad \boxed{T^{\alpha\beta}{}_{;\beta} = 0}$$

4 hlutafleiðujöfnur fyrir $g^{\alpha\beta}$-in

varðveisla orku og skriðþunga

$\Rightarrow$ 6 óháðar sviðsjöfnur fyrir 10 óþekkt föll $g^{\alpha\beta}$

með 9990 liðum (flestir eru 0)

$\rightarrow$ Kvarðafrelsi : 4 skilyrði/skorður á 1. afleiður $g^{\alpha\beta}$-anna

$$\boxed{R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}}$$

Fágatur (harmonískur) kvarði sem þekur allt tímarúmið :

$$\boxed{h^{\alpha\beta} \equiv \eta^{\alpha\beta} - (-g)^{1/2} g^{\alpha\beta}}$$

Uppfyllir skilyrði Lorentz :

$$\boxed{\sum_{\beta=0}^{3} \frac{\partial h^{\alpha\beta}}{\partial x^{\beta}} = 0}$$

SVIÐSJÖFNUR EINSTEINS:

$$\boxed{\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) h^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau^{\alpha\beta}}$$

Formleg lausn :

$$\boxed{h^{\alpha\beta}(ct, \vec{x}) = \frac{4G}{c^4} \int_C \frac{\tau^{\alpha\beta}(ct - |\vec{x}-\vec{y}|, \vec{y})\, d^3y}{|\vec{x}-\vec{y}|}}$$

RAFSEGULFRÆÐI

Fjórstraumur

$$\boxed{\underline{J} = (\underbrace{J^0}_{\rho c}, \underbrace{J^1, J^2, J^3}_{\vec{J}})}$$

Fjórmætti

$$\boxed{\underline{A} = (\underbrace{A^0}_{\frac{1}{c}V \leftrightarrow \Phi}, \underbrace{A^1, A^2, A^3}_{\vec{A}})}$$

Lorentz-skilyrði: $$\boxed{\sum_{\alpha=0}^{3} \frac{\partial A^\alpha}{\partial x^\alpha} = 0}$$ $\partial_\mu A^\mu = 0$

+

Jöfnur Maxwells

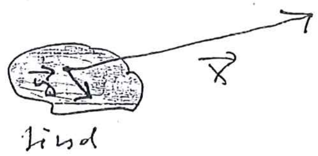
$\Downarrow$

Bylgjujafnan: $$\boxed{\underbrace{\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\Box} A^\alpha = -\mu J^\alpha}$$

Formleg lausn á bylgjujöfnunni

$$\boxed{A^\alpha(ct, \vec{x}) = \frac{\mu}{4\pi} \int_{\text{lind}} \frac{J^\alpha(ct - |\vec{x} - \vec{y}|, \vec{y})\, d^3y}{|\vec{x} - \vec{y}|}}$$

fortíðarkeila



Þægileg framsetning á $h_{\mu\nu}$

Lítum á $h_{\mu\nu}$ sem þinasvið í Minkowskirúmi. Veljum hnit í ákveðnu föstu tregðukerfi:

$$(t, x^1, x^2, x^3), \quad c = 1$$

Þá má skilgreina skalarsviðin Φ og Ψ, vigursviðið $\vec{w}$ og sporlausan þin (3×3 fylki) s_{ij} með:

$$h_{00} = -2\Phi$$
$$h_{0i} = w_i$$
$$h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$$
$$\Psi = -\frac{1}{6}\delta^{ij}h_{ij}$$
$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right)$$

Þannig má skrifa:

$$h_{\mu\nu} = \begin{pmatrix} -2\Phi & w_1 & w_2 & w_3 \\ w_1 & 2(s_{11}-\Psi) & 2s_{12} & 2s_{13} \\ w_2 & 2s_{21} & 2(s_{22}-\Psi) & 2s_{23} \\ w_3 & 2s_{31} & 2s_{32} & 2(s_{33}-\Psi) \end{pmatrix}$$

og firðina sem

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$
$$= -(1+2\Phi)dt^2 + w_i(dt\,dx^i + dx^i dt) + \left[(1-2\Psi)\delta_{ij} + 2s_{ij}\right]dx^i dx^j$$

Þetta nægir til þess að reikna Christoffel táknin

$$\Gamma^\mu_{\rho\sigma}$$

Skoðum gagnvegajöfnuna fyrir fjórskriðþungann

$$p^\mu = \frac{dx^\mu}{d\lambda}$$

($\lambda = \tau/m$ ef ögnin hefur massa m; athugið að $p^0 = \frac{dt}{d\lambda} = E$ og $E = \gamma m$; einnig er $p^i = E v^i$)

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}p^\rho p^\sigma = 0$$

$$\Downarrow$$

$\mu = 0$ jafnan:

$$\frac{dE}{dt} = -E\left[\partial_0\Phi + 2(\partial_k\Phi)v^k - \left(\partial_{(j}w_{k)} - \frac{1}{2}\partial_0 h_{jk}\right)v^j v^k\right]$$

$\mu = i$ jafnan:

$$\frac{dp^i}{dt} = E\Big[\underbrace{E_G^i + (\vec{v}\times\vec{B}_G)^i}_{\text{„Lorentz-kraftliður“}} - 2(\partial_0 h_{ij})v^j - \left(\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk}\right)v^j v^k\Big]$$

$$\downarrow$$

Gravity Probe B
$$\begin{cases} E_G^i = -\partial_i\Phi - \partial_0 w_i \iff \vec{E}_G = -\left(\nabla\Phi + \frac{\partial\vec{w}}{\partial t}\right) \\ B_G^i = (\nabla\times\vec{w})^i \iff \vec{B}_G = \vec{\nabla}\times\vec{w} \\ \text{Svarar til } \vec{E} \text{ og } \vec{B} \text{ í rafsegulfræðinni} \end{cases}$$

Þverkvarðinn (transverse gauge)

Skilyrði: $\partial_i s^{ij} = 0$ og $\partial_i w^i = 0$

$\longrightarrow$ línuleg nálgun á sviðsjöfnum Einsteins

$\left(G^{\mu\nu} = \varkappa T^{\mu\nu} \;;\; \varkappa = 8\pi G/c^4 \right)$

00-þáttur: $2\nabla^2 \Psi = \varkappa T_{00}$

0j-þáttur: $-\frac{1}{2}\nabla^2 w_j + 2\partial_0 \partial_j \Psi = \varkappa T_{0j}$

ij-þáttur: $(\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \partial_0 \partial_{(i} w_{j)} + 2\delta_{ij}\partial_0^2 \Psi - \Box s_{ij} = \varkappa T_{ij}$

d'Alembert

$\left(\Box = \nabla^2 - \frac{\partial^2}{\partial t^2}\right)$

Lorentz-skilyrði:

$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h = 0$

$$\boxed{\Box h_{\mu\nu} = -2\varkappa T_{\mu\nu}}$$

Bylgjujafnan með uppsprettum (sources)

Í tómi er $T_{\mu\nu} = 0$.
Langt frá öllum uppsprettum gildir:
Fyrir þyngdarbylgjureikninga er gagnlegt að 4. skilyrðið sé hið svokallaða „sporlaus-skilyrði“
$\nabla^2\Phi = \nabla^2\Psi = \nabla^2 w_i = 0$ sem með eðlilegum jaðarskilyrðum leiðir til $\Phi = \Psi = w_i = 0$

Í þessum kvarða, sem er kallaður sporlausi þverkvarðinn (transverse-traceless gauge = TT), $h^{TT}_{\mu\nu}$ er

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2s_{11} & 2s_{12} & 2s_{13} \\ 0 & 2s_{21} & 2s_{22} & 2s_{23} \\ 0 & 2s_{31} & 2s_{32} & 2s_{33} \end{pmatrix}$$

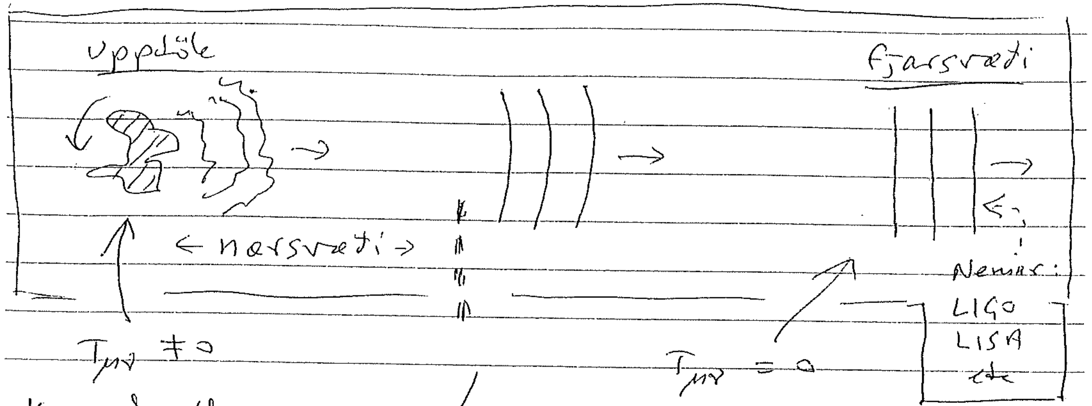
Samhverfur og sporlaus þáttur

$(s_{ij} = s_{ji})$ $(s_{ii} = 0)$

Bylgjujafnan í tómi langt frá öllum uppsprettum verður þá

$$\Box h^{TT}_{\mu\nu} = 0$$

Þyngdargeislun



Kannað síðar:

I_{ij}

J_{ij}

orkubúskapur og geislunarafl

Hér má nota TT-hnitakerfi:

$$h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2S_{11} & 2S_{12} & 2S_{13} \\ 0 & 2S_{21} & 2S_{22} & 2S_{23} \\ 0 & 2S_{31} & 2S_{32} & 2S_{33} \end{pmatrix}$$

samhverfur og sporlaus

Bylgjujafnan hér

$$\boxed{\Box h_{\mu\nu}^{TT} = 0}$$

Lausnir á formi sléttra bylgna

$$\boxed{h_{\mu\nu}^{TT} = C_{\mu\nu}\, e^{i k_\alpha x^\alpha}}$$

$C_{\mu\nu}$: þinur 4x4 fylki með föstum stökum

k_α: fasti

Lausnir á $\Box h_{\mu\nu}^{TT} = 0$

$$h_{\mu\nu}^{TT} = C_{\mu\nu}\, e^{i k_\alpha x^\alpha}$$

Almenna lausnin er línuleg samantekt af svona planbylgjum

(a) Innsetning í bylgjujöfnuna $(\Box h_{\mu\nu}^{TT} = 0)$ gefur

$$\Box h_{\mu\nu}^{TT} = -\underbrace{k_\alpha k^\alpha}_{\text{innfeldi } \bar{k} \text{ (fjórvigursins)}} h_{\mu\nu}^{TT} = 0$$

$\Rightarrow$ $k_\alpha k^\alpha = 0$ svo bylgjuvigurinn $\bar{k}$ er núllvigur $\Rightarrow$ Hraði bylgjunnar er ljóshraðinn $= c = 1$

(b) Skoðum næst summuna $k_\alpha x^\alpha$:

$$k_\alpha x^\alpha = k_0 ct + \underbrace{k_1 x^1 + k_2 x^2 + k_3 x^3}_{\vec{k}\cdot\vec{r}}, \quad \vec{r} = (x^1, x^2, x^3)$$

Nú er $\omega = k^0 c$ (vigur) $= -k_0 c$ (þakvigur)

$\Rightarrow$ $k_0 c = -\omega$

$\Rightarrow$ $\boxed{k_\alpha x^\alpha = -\omega t + \vec{k}\cdot\vec{r}}$

Fjórvigurinn er því $\bar{k} = (k^0, \vec{k}) = \left(\frac{\omega}{c}, \vec{k}\right)$

c) Þverkvarða-skilyrðið $\partial_\mu \bar{h}^{\mu\nu} = 0$ leiðir til

$$\partial_\mu h^{\mu\nu}_{TT} = \underbrace{i\, C_{\mu\nu}\, e^{i k_\sigma x^\sigma}\, k_\mu}_{} = 0$$

$$\hookrightarrow \quad k_\mu C^{\mu\nu} = 0$$

Veljum nú viðmiðunarásana í fjarsviðinu þ.e. bylgjan hreyfist í z-stefnu, þ.e.

$$\vec{k} = (0,0,k)$$

Þá er $\bar{k} = \left(\frac{\omega}{c}, 0, 0, k\right) = \left(\frac{\omega}{c}, 0, 0, \frac{\omega}{c}\right)$ þar sem $\omega = kc$

og $k_\mu C^{\mu\nu} = k_0 \underbrace{C^{0\nu}}_{=0} + \cancel{k_1} C^{1\nu} + \cancel{k_2} C^{2\nu} + k_3 C^{3\nu}$

$= k_3 C^{3\nu} = 0 \Longrightarrow C^{3\nu} = 0$

$$\Longrightarrow \quad C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

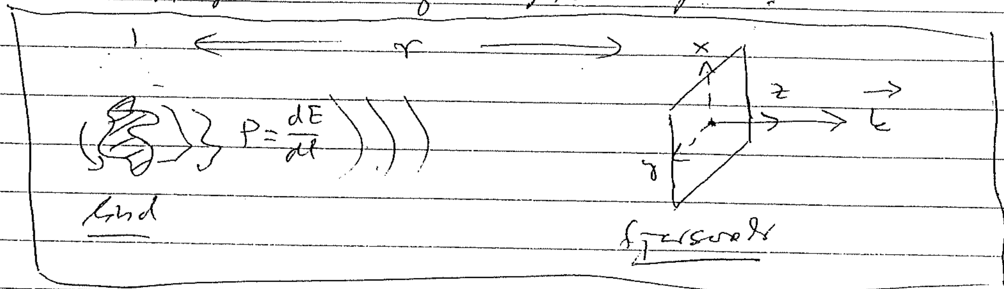
þar sem δ_{ij} og þ.e.l. C_{ij} er samhverfur og sporlaus

Venjan er að gefa þessum stökum ný nöfn (sem verða útskýrð seinna), nefnilega $C_{11} = h_+$ og $C_{12} = h_\times$

Þannig að lausnina má skrifa sem:

$$\boxed{h^{TT}_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\omega t - kz)}}$$

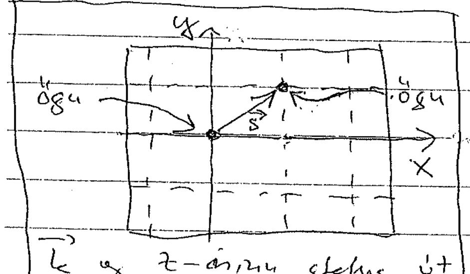
Stuðlarnir h_+ og $h_\times$ eru fastar á hverjum stað og gildi þeirra ákvarðast af fjarlægðinni til uppsprettunnar og útgeislunaraflsins:



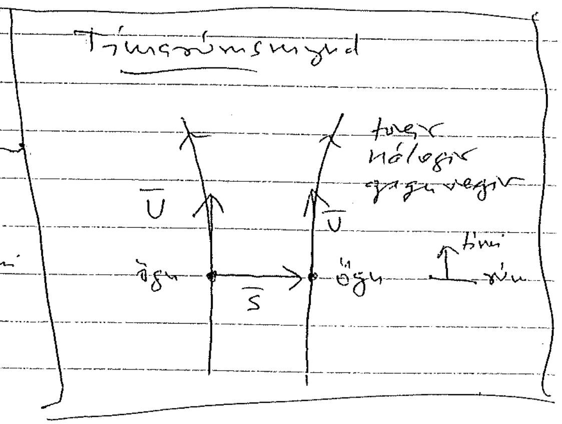
Bylgjan er þverstæð planbylgja í fjarsviðinu: Einar sveiflurnar verða í xy-planinu.

Lítum nánar á slíkar sveiflur:

Byrjum á því að leggja samskipta hnitanet þ.e. hver ögn hefur sín föstu hnit. Skoðum tvær punktagnir og látum eiginfjarlægðina milli þeirra vera $\bar{S}$:



$\vec{k}$ og z-ásinn stefna út



$\bar{U} = (1,0,0,0)$

Lítill þáttur, mjög stuttur tími

Gagnvegs-vikjöfnun fyrir agnirnar [illegible]

$$\boxed{\frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma}$$

Önnur kóvaríant afleiða

Í 1. gráðu nálgun er $\frac{D^2}{d\tau^2} \approx \frac{\partial^2}{\partial t^2}$

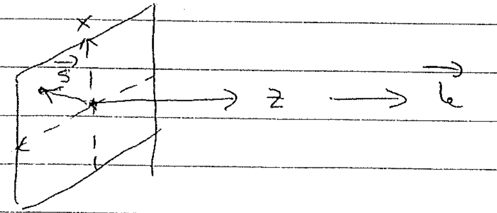
og í útreikningum þurfum við aðeins þætti með $\nu = \rho = 0$:

$\Rightarrow R_{\mu 0 0 \sigma} =$ talsverðir reikningar $\ldots = \frac{1}{2}\frac{\partial^2 h^{TT}_{\mu\sigma}}{\partial t^2}$

og

$$\boxed{\frac{\partial^2 S^\mu}{\partial t^2} \approx \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} h^{TT\,\mu}{}_\sigma\right) S^\sigma}$$

Athugið að aðeins S^1 og S^2 hnitin í þrívigrinum $\vec{S} = (S^1, S^2, S^3)$ breytast. S^3 er fasti og við getum sett $S^3 = 0$ (þar sem $h^{TT\,3}{}_\sigma = 0$).



Í reikningum er þægilegt að velja ~~hnitakerfið þ.a.~~ $z = 0$ í annarri ögninni

1. Skoðum fyrst svokallaða + - skautun ($h_\times = 0$)

Þá verða vikjöfnurnar

$$\frac{\partial^2 S^1}{\partial t^2} = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2}\left(h_+ e^{-i(\omega t - k\cancel{z}^{0})}\right)$$

$$\frac{\partial^2 S^2}{\partial t^2} = -\frac{1}{2} S^2 \frac{\partial^2}{\partial t^2}\left(h_+ e^{-i(\omega t - k\cancel{z}^{0})}\right)$$

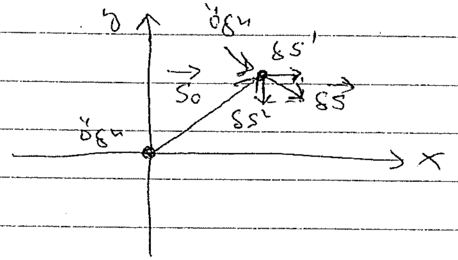
Fyrstu-gráðu lausnir á þessum jöfnum eru

$$S^1(t) = S_0^1\left(1 + \frac{1}{2} h_+ e^{-i\omega t}\right)$$
$$S^2(t) = S_0^2\left(1 - \frac{1}{2} h_+ e^{-i\omega t}\right)$$

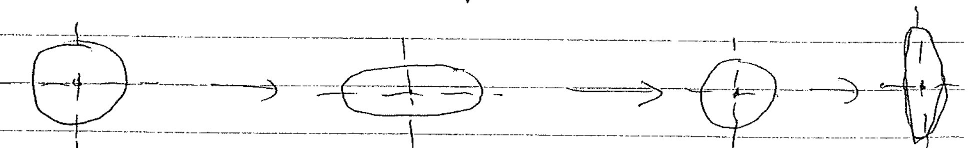
sem má umskrifa á formið

$$\delta S^1 = S^1 - S_0 = \frac{1}{2} h_+ e^{-i\omega t} S_0^1$$
$$\delta S^2 = S^2 - S_0 = -\frac{1}{2} h_+ e^{-i\omega t} S_0^2$$

Sveiflulausnir



Með því að hugsa sér agnahring umhverfis ögnina í (0,0,0) fæst nokkuð klassísk mynd af þyngdarbylgjum fyrir + skautun:



~~Sjá myndir bls. 279 í D'Inverno~~

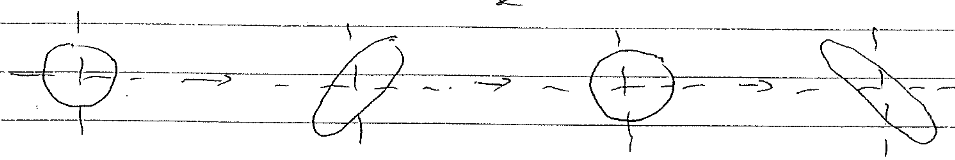
2. X-skautun $(h_+ = 0)$

Með því að beita sömu aðferðum og í 1. fást lausnirnar

$$\delta S^1 = \frac{1}{2} h_\times e^{-i\omega t} S_0^2$$

$$\delta S^2 = \frac{1}{2} h_\times e^{-i\omega t} S_0^1$$

sveiflulausnir



3. Almennt er sveiflan línuleg samantekt úr + og × sveiflunum.

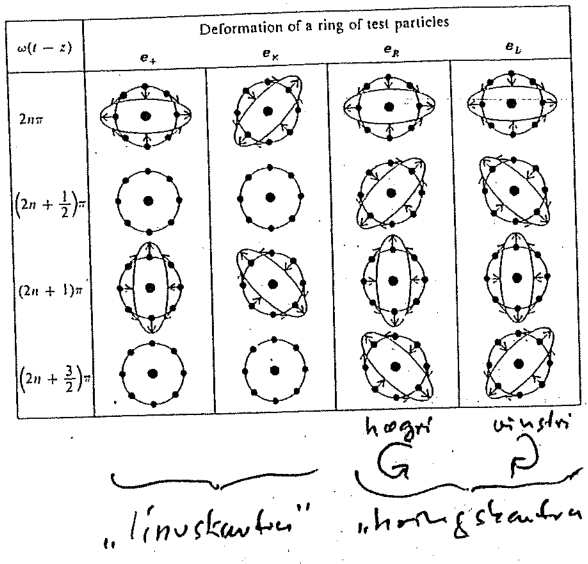


Figure 35.2.
Plane Gravitational Waves. Polarization tensor: e_P

Metric perturbation:

$$h_{jk} = \Re\,[A_0 e^{-i\omega(t-z)} e_{Pjk}]$$

Tidal acceleration between two test particles:

$$\frac{D^2 n_j}{D\tau^2} = -R_{j0k0} n_k = \frac{1}{2}\frac{\partial^2 h_{jk}}{\partial t^2} n_k = \Re\left[-\frac{1}{2}\omega^2 A_0 e^{-i\omega(t-z)} e_{Pjk} n_k\right]$$

Separation between two test particles:

$$n_j = n_j^{(0)} + \Re\left[\frac{1}{2} A_0 e^{-i\omega(t-z)} e_{Pjk} n_k^{(0)}\right]$$

Position of test particle B in proper reference frame of test particle A. (In drawing, A is the central particle and B is any peripheral particle):

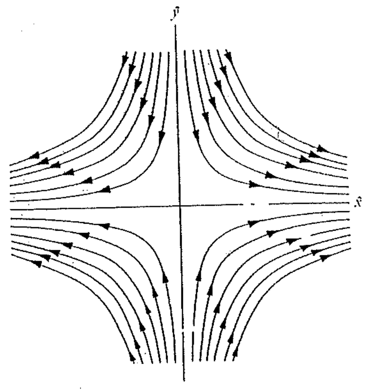
$$x_B^{\,j} = x_{B(0)}^{\,j} + \Re\left[\frac{1}{2} A_0 e^{-i\omega(t-z)} e_{Pjk} x_{B(0)}^{\,k}\right]$$

$$d^2\hat{x}/d\hat{t}^2 = \frac{1}{2}(\ddot{A}_+\hat{x} + \ddot{A}_\times\hat{y}),$$
$$d^2\hat{y}/d\hat{t}^2 = \frac{1}{2}(-\ddot{A}_+\hat{y} + \ddot{A}_\times\hat{x}), \qquad (1)$$
$$d^2\hat{z}/d\hat{t}^2 = 0.$$

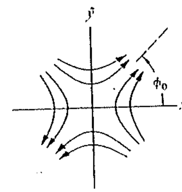
Notice that these accelerations are divergence-free. Consequently they can be represented by "lines of force," analogous to those of a vacuum electric field. At a value of $\hat{t} - \hat{z}$ where $\ddot{A}_\times = 0$ (so polarization is entirely e_+), the lines of force are the hyperbolas shown here [sketch (a)]. The direction of the acceleration at any point is the direction of the arrow there; the magnitude of the acceleration is the density of force lines. Since acceleration is proportional to distance from center of mass, the force lines get twice as close together when one moves twice as far away from the origin in a given direction. When $\ddot{A}_+$ is positive, the arrows on the force lines are as shown in (a); when it is negative, they are reversed. As $|\ddot{A}_+|$ increases, the force lines move in toward the origin so their density goes up; as $|\ddot{A}_+|$ decreases, they move out toward infinity so their density goes down.

For polarization $e_\times$ the force lines are rotated by 45° from the above diagram. For intermediate polarization (values of $\hat{t} - \hat{z}$ where $\ddot{A}_+$ and $\ddot{A}_\times$ are both nonzero), the diagram is rotated by an intermediate angle [sketch (b)]

$$\phi_0 = \frac{1}{2}\arctan(\ddot{A}_\times/\ddot{A}_+). \qquad (2)$$



(a) Force lines for $\ddot{A}_\times = 0$, $\ddot{A}_+ > 0$



(b)

Upptök þyngdarbylgna:

Í upptökunum er $T_{\mu\nu} \neq 0$ og því ekki hægt að nota einfalda aðferð eins og að velja TT-hnita og vonast til að reikningar verði einfaldir. Truflanir eru venjulega flóknar og þá verður að beita tölvulegum reikningum, sem oft eru reyndar nokkuð ofviða.

Ef breytingar í þyngdarsviði uppsprettunnar eru „tiltölulega rólegar", hraðinn ekki mjög mikill og „árekstrar" eiga sér ekki stað, kemur í ljós að venjulega má byrja með upphaflega „þinnandi" $h_{\mu\nu}$ í Minkowski rúmi (sjá kb. 5 í 1. hluta) og gera einfalda hnitabreytingu:

$$h_{\mu\nu} \to \bar{h}_{\mu\nu}$$

þar sem

$$\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}$$

Hér er $h = \eta^{\mu\nu} h_{\mu\nu}$ og $\bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = -h$

Þessi nýja hnit er gagnlegur vegna þess að á fjarsviði er auðvelt að breyta honum í sporlausan þverhnita $\bar{h}^{TT}_{\mu\nu}$ og í ljós kemur að þar gildir

$$\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}$$

lausnirnar sem raktar voru í 2. hluta

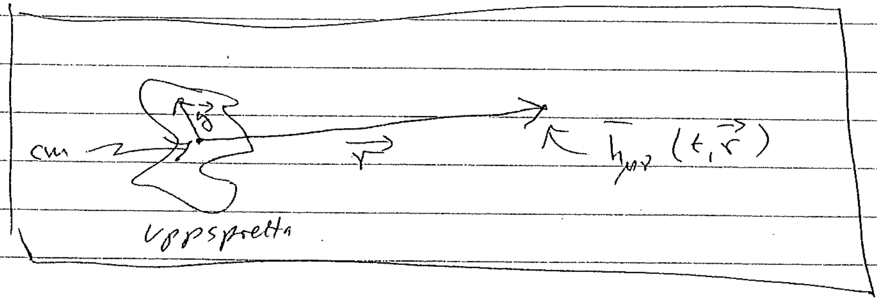
Í þessum hnitum eru sviðsjöfnur Einsteins á forminu

$$\boxed{\Box \bar{h}_{\mu\nu} = -2\underbrace{\kappa}_{(8\pi G/c^4)} T_{\mu\nu}}$$

Seinkaða lausnin er (sjá rafsegulfræðina):

$$\boxed{\bar{h}_{\mu\nu}(t,\vec{r}) = \frac{4G}{c^4} \underbrace{\int}_{\text{Rúmtaksheild uppsprettu}} \frac{T_{\mu\nu}(t_r,\vec{y})\, d^3y}{|\vec{r}-\vec{y}|}}$$

Hér er $t_r = t - |\vec{r}-\vec{y}|$ og $\vec{y}$ staðsetur punkt í uppsprettunni:



Á fjarsviði er $|\vec{y}| \ll |\vec{r}| = r$

Ef $T_{\mu\nu}$ breytist ekki kröftuglega má því setja $|\vec{r}-\vec{y}| \approx r$

og
$$\boxed{\bar{h}_{\mu\nu}(t,\vec{r}) = \frac{4G}{c^4 r} \underbrace{\int}_{\text{Ind = uppspretta}} T_{\mu\nu}(t_r,\vec{y})\, d^3y}$$

á fjarsviði

$T_{\mu\nu}$ fyrir uppsprettuna

Almennt má skrifa í þverstöðum Lorentz-grunni (í Minkowski rúmum):

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & P_1 & P_2 & P_3 \\ P_1 & T_{11} & T_{12} & T_{13} \\ P_2 & T_{21} & T_{22} & T_{23} \\ P_3 & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

ρ = þéttleiki massa og orku

P_i = orku- og ~~skriðþungaflæði~~ í stefnu x_i

$$= \frac{F_i}{c} = \frac{1}{c}\left(\frac{dE/dt}{A}\right)_i$$

$T_{ij} = P$ öll ij þar sem P er þrýstingur
Gildir í stefnuháðu efni

spennuþinur

Varðveisla skriðþunga og orku:

$$\partial_\nu T^{\mu\nu} = 0 \implies \int_{\text{lind}} T_{0\mu}\, d^3y = \text{fasti}$$

Heildarorka og heildarskriðþungi eru varðveittar stærðir.

$\implies$ $\bar{h}_{0\mu}$ er óháð tíma (fyrir öll μ) svo sveiflurnar eru eingöngu í $\bar{h}_{ij}$

Fjórpólsþinur efnisdreifingarinnar

(quadrupole tensor)

Hann er skilgreindur með

$$\boxed{I_{ij}(t) = \frac{1}{c^2}\int_{\text{lind}} y_i y_j T^{00}(t,\vec{y})\, d^3y = \int_{\text{lind}} y_i y_j\, \rho(t,\vec{y})\, d^3y}$$

2-þinur
3×3 fylki

Með umtalsverðum reikningum má sýna fram á að lausnin fyrir $\bar{h}_{ij}(t,\vec{r})$ er gefin með

$$\boxed{\bar{h}_{ij}(t,\vec{r}) = \frac{2G}{c^4 r}\frac{d^2}{dt^2} I_{ij}(t_r)}$$

gildir fyrir „rólegt upphaf" eins og tvístirni t.d. NS+NS eð SH+NS eð SH+SH meðan tvístirnin lenda ekki í árekstri

Formúlan gildir einungis í fjarsviði og því er hagkvæmt að fara yfir í h^{TT}_{ij} kvarða með

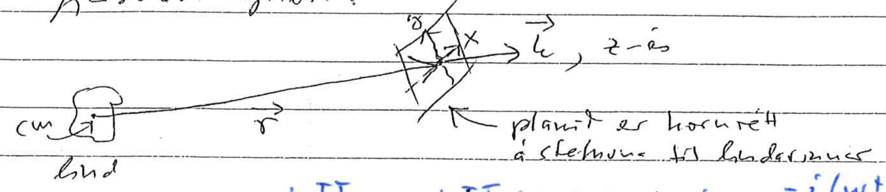
$$h^{TT}_{ij} = \bar{h}^{TT}_{ij} \Longleftarrow \bar{h}_{ij}$$

Það er gert með því að skilgreina „sporlausa fjórpólsþininn" (reduced quadrupole tensor) $J_{ij}(t_r)$

-5-

$$\boxed{J_{ij}(t_r) = I_{ij}(t_r) - \frac{1}{3}\delta_{ij}\,\delta^{kl} I_{kl}(t_r)}$$

Fyrir bylgju sem hreyfist í z-stefnu á fjarsviði gildir:



$$h_+^{TT} = h_+^{TT}(t,z) = h_+ e^{-i(\omega t - kz)} \text{ etc}$$

$$h_+^{TT} = h_{xx}^{TT} = -h_{yy}^{TT} = \frac{G}{c^4 r}\left\{\frac{d^2 J_{xx}}{dt^2} - \frac{d^2 J_{yy}}{dt^2}\right\}$$

$$h_\times^{TT} = h_{xy}^{TT} = h_{yx}^{TT} = \frac{2G}{c^4 r}\cdot\frac{d^2 J_{xy}}{dt^2}$$

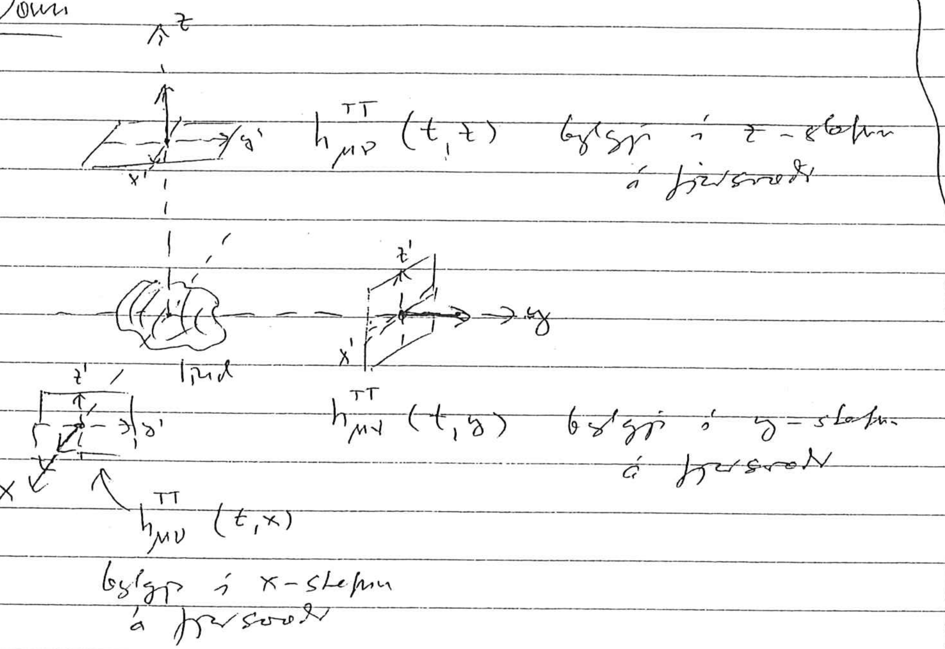
og

$$h_{\mu\nu}^{TT}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i\omega(t-z)}$$

(munið: $\omega = kc$)

-6-

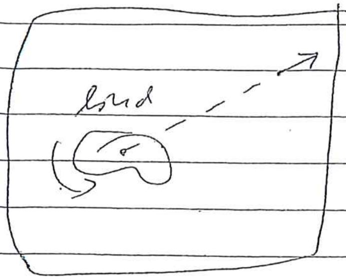
Dæmi



Á myndinni má sjá að til að finna geislun í aðra stefnu en z má snúa hnitakerfi þ.a. geislun í x-stefnu fæst með því að nota stærðir með vísum x í stað z
y í stað x
z í stað y
(í formúlum)

Fyrir y-stefnu kemur y í stað z
z í stað x
x í stað y

Orkubúskapur



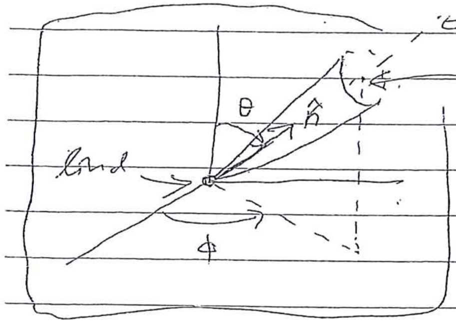
Orkuflæði í ákveðna stefnu ($\hat n$) er gefið með

$$\left[\frac{1}{A}\left(\frac{dE}{dt}\right)\right]_{\hat n} = F_i = c\,p_i = c\,T_{0i}$$

Umtalsverðir reikningar sýna eftirfarandi

$$\left[\frac{1}{A}\left(\frac{dE}{dt}\right)\right]_{\hat n} = \cdots = \frac{c^4}{32\pi G}\left\langle \partial_0 h_{jk}^{TT} \cdot \partial_i h^{TT\,jk}\right\rangle$$

Með því að nota niðurstöðurnar fyrir h_{jk}^{TT} úr fyrri ~~kafla~~ kafla og gefa stefnu til lengra með einingarvigrinum $\hat n$ fæst



$$d\Omega = \sin\theta\, d\theta\, d\phi$$

$$\hat n = \frac{\vec r}{|\vec r|} = \frac{\vec r}{r} = (n^1, n^2, n^3)$$

Orkuþéttni í stefnu $\hat n$: $$\frac{d^2E}{dt\,d\Omega} = \frac{G}{16\pi r^2 c^5}\left\langle 2\frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3} - 4n^k\frac{d^3J_{kj}}{dt^3}\frac{d^3J^{j}{}_{l}}{dt^3}n^l + \left(n^in^j\frac{d^3J_{ij}}{dt^3}\right)^2\right\rangle$$

Með því að heilda yfir heilu fæst:

$$\boxed{L_{GW}(t) = \frac{dE}{dt} = \int_{4\pi}\left(\frac{d^2E}{dt\,d\Omega}\right) r^2\, d\Omega = \frac{G}{5c^5}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3}\right\rangle}$$

gildi reiknuð á seinkaðan tímann t_r

Mjög veik svið : $\epsilon \ll 1$

$|h_{ij}| \ll 1$

$$\Rightarrow \begin{cases} g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} \\ \Lambda^{\mu\nu} \simeq 0 \quad \text{og} \quad \tau_{\mu\nu} \approx T_{\mu\nu} \end{cases}$$

$$\Rightarrow \quad h_{ij}(ct, \vec x) \simeq \frac{2G}{c^4 r}\ddot{I}_{ij}(ct - r)$$

Orkutap lindar vegna þyngdargeislunar í þessari nálgun er

$$\boxed{\frac{dE}{dt} \simeq -\frac{G}{5c^5}\left\langle \sum_{ij} \dddot{\bar I}_{ij}\, \dddot{\bar I}^{ij}\right\rangle}$$

nokkrar bylgjulengdir

$$\sim -\underbrace{\left(\frac{c^5}{G}\right)}_{=\,3{,}6\times 10^{52}\,\mathrm{W}}\left(\frac{R_s}{y}\right)^2\left(\frac{v}{c}\right)^6$$

$$J_{ij} = \bar I_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I^k{}_k$$

ÞYNGDARGEISLUN

VIÐFANGSEFNI

Lindir: Gerð og útgeislun

- stærðfræðileg umfjöllun
- tölulegar reikningar

- stjörnufræðilegar mælingar
- hátækni, verkfræði
- gagnaúrvinnsla

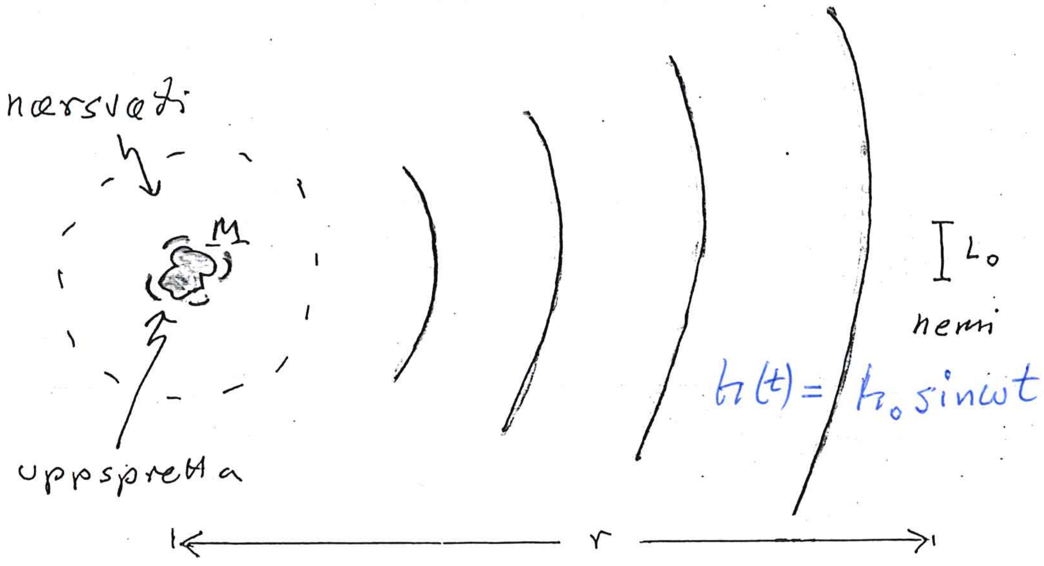
Bylgjuútbreiðsla
Áhrif á stjörnufræðileg kerfi – Heimsfræði, t.d. örbylgjukliður

- stærðfræðileg umfjöllun
- tölulegar reikningar

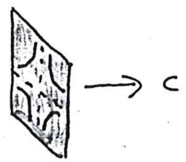
Beinar mælingar á sólkerfinu:
LIGO, LISA, ...

- tilraunaeðlisfræði
- hátækni, verkfræði
- gagnaúrvinnsla
- tölulegar reikningar.

Þyngdarbylgjur



fjarsvæði
↕
<u>sléttar bylgjur</u>



$$\boxed{\frac{\Delta L}{L_0} = \frac{1}{2}h = \frac{1}{2}h_0 \sin(\omega t + \phi)}$$

<u>Fjórpólsnálgun</u>

$$\boxed{h_0 \sim \frac{2GM}{c^2 r}\left(\frac{v^2}{c^2}\right)}$$

$v^2 \sim$ meðalvarði á hreyfiorku sem ekki er í kúlusamhverfri hreyfingu

<u>Orkuflæði</u>

$$\boxed{\langle F\rangle \simeq \frac{1}{64\pi}\frac{c^3\omega^2 h_0^2}{G}} \longleftrightarrow \boxed{\langle F\rangle = \frac{1}{A}\left\langle\frac{dE}{dt}\right\rangle}$$

$$h_0 \sim \frac{2GM}{c^2 r}\left(\frac{v^2}{c^2}\right)$$

$$\frac{2GM}{c^2 r} \simeq 5\times10^{-21}\left(\frac{M}{M_\odot}\right)\left(\frac{r}{20\,\mathrm{Mpc}}\right)^{-1}$$

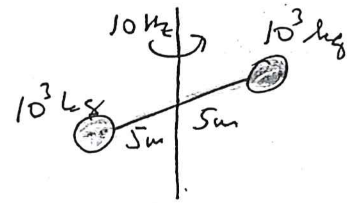
fjarlægð til Virgo hópsins

<u>Áhrif á nema</u>:

$$\left.\begin{aligned}(\Delta L)_{max} &= \frac{h_0 L_0}{2}\\ L_0 &\sim 1\ \mathrm{km}\\ h_0 &\sim 10^{-22}\end{aligned}\right\} \Rightarrow (\Delta L)_{max} \sim 10^{-19}\ \mathrm{m}$$

$\sim 10^{-4}\cdot$ (radíus vetniskjarna)

<u>Þyngdarbylgjur í rannsóknastofu</u>



$v^2 \simeq 10^5\ \mathrm{m^2/s^2}$

$\lambda \simeq 1{,}5\times10^4\ \mathrm{km}$ ($f = 20\ \mathrm{Hz}$)

„sama staða" þrisvar á um fastan tíma

og

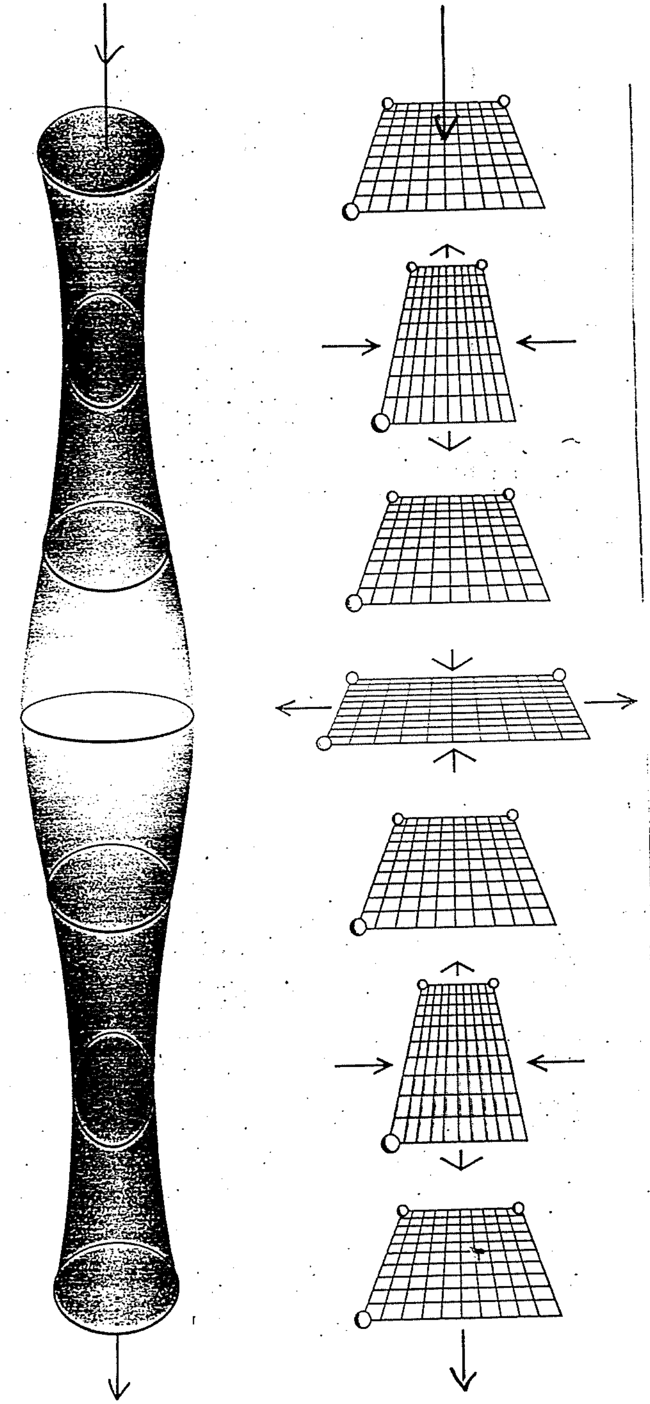
$h_0 \sim 2\times10^{-43}$

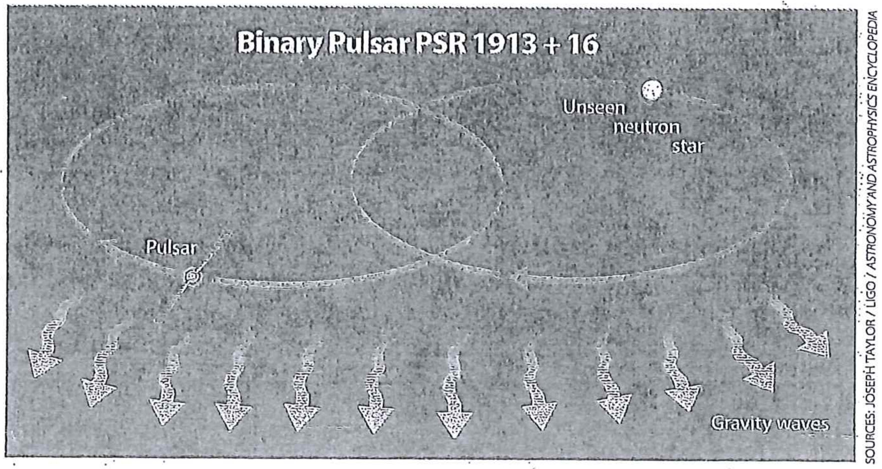
1/3 '01
EHG

$$\langle F \rangle \approx \frac{1}{64\pi}\,\frac{c^3 \omega^2 h_0^2}{G} \approx 10^{-5} \left(\frac{f}{100\,\mathrm{Hz}}\right)^2 \left(\frac{h_0}{10^{-22}}\right)^2 \frac{\mathrm{W}}{\mathrm{m}^2}$$

Til samanburðar : Ljósflæði frá Júpíter hér á jörðu

$$\langle F \rangle = 2{,}3 \times 10^{-7}\ \mathrm{W/m^2}$$





SOURCES: JOSEPH TAYLOR / LIGO / ASTRONOMY AND ASTROPHYSICS ENCYCLOPEDIA

Umferðartími: $7{,}75^h$

Brautarsnúningur: $4{,}23^\circ$

Tiftími: 59 ms

M: $\sim 1{,}4\, M_\odot \times 2$

$$\Delta t = -\left(\frac{\dot{\omega}}{2\omega}\right) t^2$$

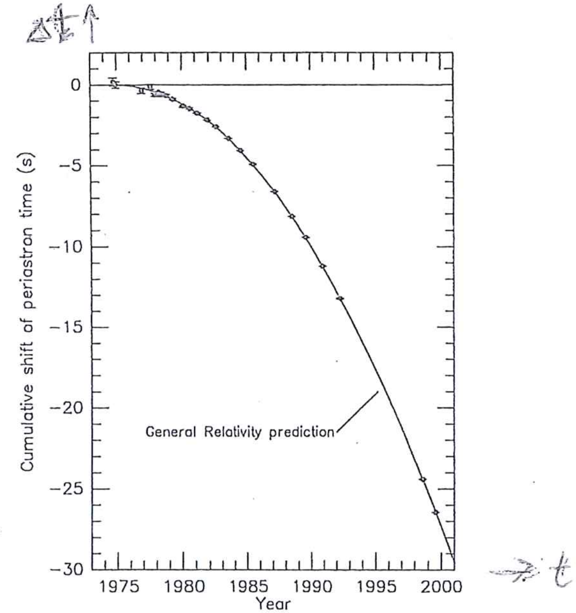
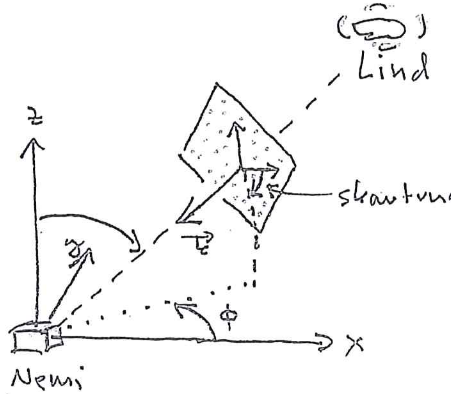


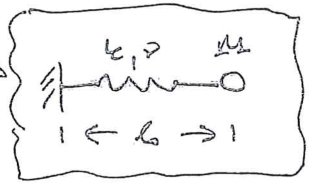
Figure 7: Plot of cumulative shift of the periastron time from 1975 – 2000. Points are data, curve is the GR prediction. Gap during the middle 1990s was caused by closure of Arecibo for upgrading. [J. H. Taylor and J. M. Weisberg, 2000, private communication].

Í samræmi við AA

Nemar



þyngdarbylgjur:
$h \sim \frac{1}{r}$



$l(t) = l_0 + \xi(t)$
eginlengd

Þvingaður og dempaður sveiffall:

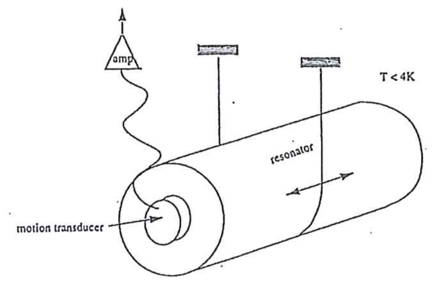
$$\boxed{\ddot{\xi} + 2\gamma\dot{\xi} + \omega_0^2\xi = \frac{1}{2} l_0 f(\theta,\phi,\psi,t)}$$

$$f = F_+(\theta,\phi,\psi)\ddot{h}_+(t) + F_\times(\theta,\phi,\psi)\ddot{h}_\times(t)$$

mynsturføll

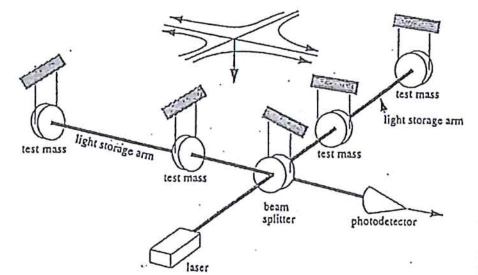
$$\boxed{\begin{aligned}\omega_0^2 &= \frac{2k}{m}\\ 2\gamma &= \frac{2\nu}{m} = \frac{\omega_0}{Q}\end{aligned}}$$

Sívalningsnemi

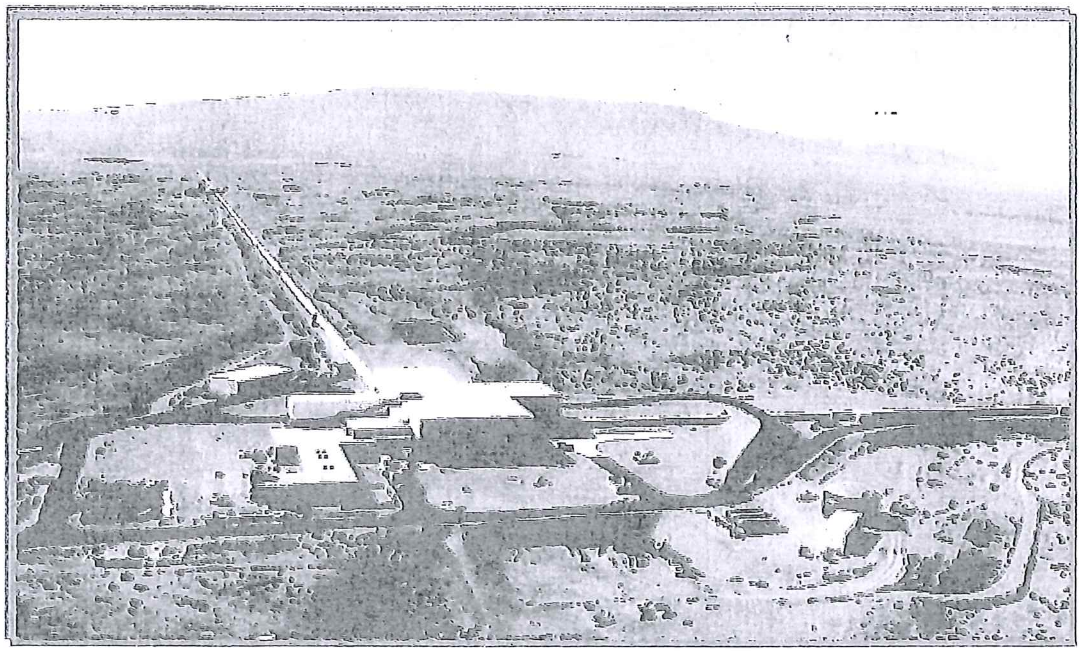


Nemi:
$h \sim 5\times10^{-19}$
við 1 kHz

Víxlunarnemi



Nemi:
$h \sim 10^{-23}$
við
$\sim 10^{2}$ Hz LIGO
$\sim 10^{-2}$ Hz LISA



LIGO - HANFORD

Armar : 4 km hvor

Næmi : ~ þvermál róteindar / 1000

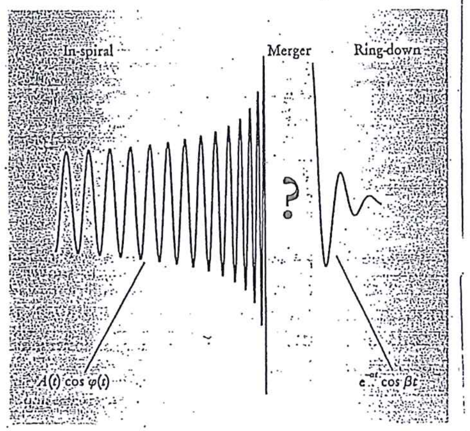


FIGURE 5. GRAVITATIONAL WAVEFORM expected from compact binary in-spiral, merger, and ring-down of a final black hole. During the in-spiral phase, a post-Newtonian approximation carried to high powers of v/c beyond Newtonian order accurately describes the orbit and waveform, with amplitude $A(t)$ and phase $\varphi(t)$ that evolve nonlinearly with time. The merger waveform is unknown at present; to determine it is the primary goal of numerical relativity. The ring-down waveform is a superposition of damped normal modes. For each mode, the damping coefficient α and frequency β have been thoroughly calculated by means of perturbation theory and have been cataloged as functions of the mass and spin of the black hole.

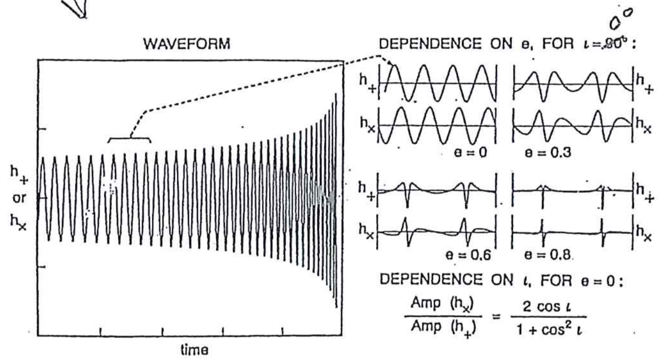
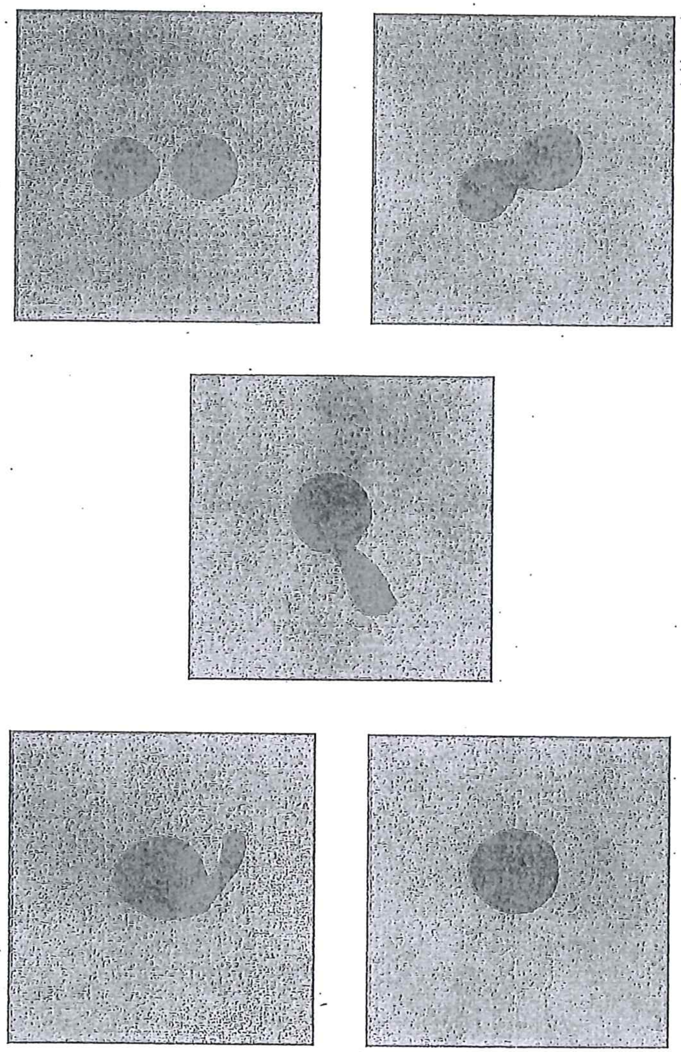


Figure 7: Waveforms from the inspiral of a compact binary, computed using Newtonian gravity for the orbital evolution and the quadrupole-moment approximation for the wave generation. (From Ref. [11].)

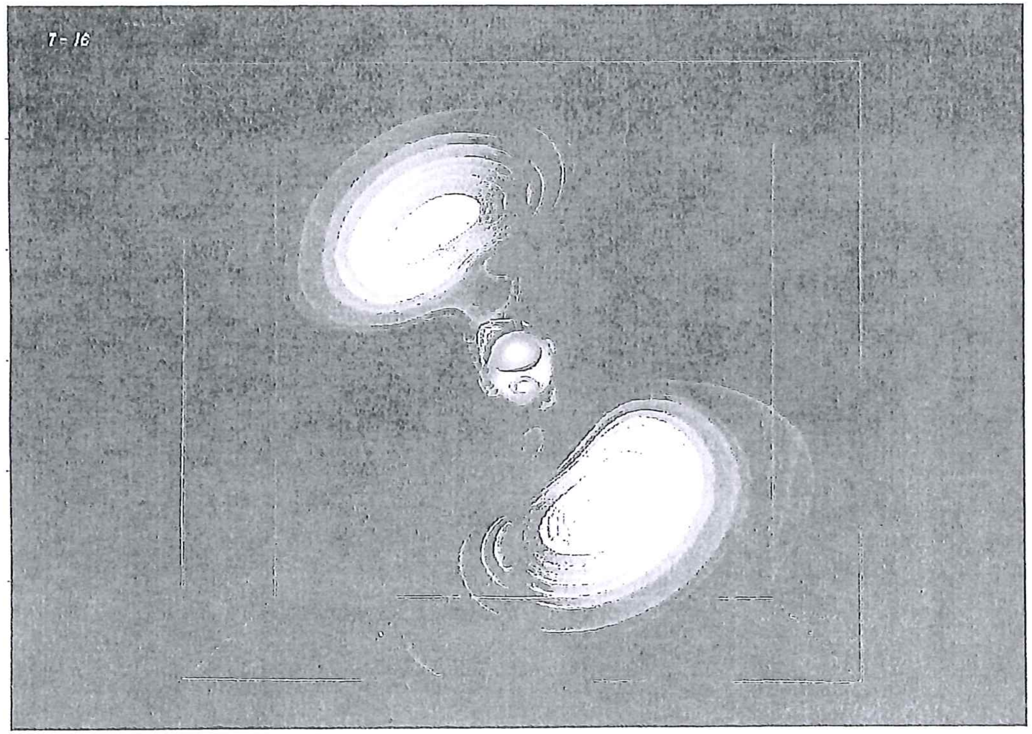
SAMRUNI

TVEGGJA NIFTEINDASTJARNA



Árekstur tveggja svarthola

$10\,M_\odot + 15\,M_\odot$



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