

ROYAUMONT – PROPOSALS ON ARITHMETIC AND ALGEBRA TEACHING FOR LOWER-SECONDARY SCHOOL LEVEL

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This study investigates influences of the Royaumont meeting on arithmetic and algebra textbooks at lower-secondary school level in the Nordic countries, Iceland in particular. Nordic and Icelandic New-Math textbooks are compared to earlier textbooks and more recent ones, in introduction of set theoretical concepts and notation, structure of the number field, and the study of numbers. Set theoretical concepts and notation dwindled rapidly, while the structure of the number field and in particular the study of numbers, received a more permanent place in the syllabus than earlier. New topics, such as coordinate geometry, probability and statistics, were devoted increased attention.

INTRODUCTION

In the post-WWII era, questions arose in many countries about school mathematics. An international reform movement in mathematics education had at least three origins: the New-Math movement in the United States of America, the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM) in French-speaking Europe, and reform plans in Britain. The different reform movements met at Royaumont, France, in November 1959 at a seminar for mathematicians, mathematics educators and mathematics teachers on reform of school mathematics, organized by the OEEC. Waves from that meetings spread around the world. The Nordic countries joined in collaboration on curriculum and textbook development. The purpose of this paper is to analyse changes in the content of arithmetic and algebra textbooks at lower secondary level in Iceland, as one of the Nordic countries. The analysis concerns three topics: introduction of set theoretical concepts and notation, structure of the number field, and the study of numbers. The question is which of the various topics survived the first wave of enthusiasm created by the reform movements to become a permanent part of the curriculum.

PRESENTATIONS AT THE ROYAUMONT SEMINAR

Mathematics professor Gustave Choquet, chairman of the CIEAEM and guest speaker at the Royaumont meeting, spoke on *new thinking in mathematics education* (OEEC, 1961, pp. 62–68). Tendencies in modern mathematics to do away with boundaries between arithmetic, algebra, geometry and calculus could be realized through the study of structures. The sets N and Z were endowed with numerous structures such as order, group or ring. The set Z constituted an excellent basis for study. Its “discrete” character made it tangible so that it might be used for introducing and studying such concepts as one-to-one correspondence, function, conversion and equivalence.

In secondary schools one could explore the algebra of sets and its symbolism; functions; graphs; relation between algebra of sets and logic. Furthermore, the set of integers developed by means of the Peano axioms; the ring of integers, Z ; and the field of rationals, Q . From arithmetic could be taken divisibility; Euclidean algorithm; prime numbers; irrationals; Diophantine equations; and geometric representation on the plane of $Z \times Z$. Examples of a group could be explored; ring of

integers modulo n ; and homomorphisms. Choquet mentioned series of squares with its successive differences as well as of cubes, probability, and some concepts of combinatorial analysis. No square roots were to be taught, but rather calculations in the binary, octad and duo-decimal numeration. Teaching of algebra had mostly been concerned with how to do, neglecting the deductive aspects.

Willy Servais, Belgian secondary school teacher and the secretary of CIEAEM proposed reforms of algebra teaching (OEEC, 1961, pp. 22–23, 62–63, 68–73). The notions of sets, functions, Cartesian product, relations and operations were prerequisite to all others, as sets form the basis of the construction of mathematics. The properties of the algebra of sets should be discovered rather than expounded. Sets also proved a good foundation for elementary notions of logic. The use of logic and logical symbols was essential in secondary school. There was, however, a danger that pupils would assume that symbols replace thinking; they should facilitate thinking. Another member of the CIEAEM, psychologist Jean Piaget, revealed his conclusions on the similarities of the structure of the mind and mathematical structures which awoke great interest among mathematicians.

According to recommendations of the seminar (OEEC, 1961, pp. 105–125), new topics such as abstract algebra, vector spaces, theory of sets etc. were to enter the secondary-school programme. New applications of mathematics suggested new problem material, such as probability, statistical inference, finite mathematical structures, linear programming and numerical analysis. More people must be better trained in scientific knowledge. Even laymen must become to know mathematics and understand science. New mathematical ideas and new symbolism would lead to clarity, compactness and unity of mathematical exposition. Compartmentalisation was contrary to the recommendations and one of the aspects of modern mathematics was its unity. As the child processes through secondary school, linear algebra, vectors and coordinate geometry should grow into a unified body of knowledge.

THE NORDIC COMMITTEE FOR MODERNIZING MATHEMATICS TEACHING

Nordic participants at Royaumont agreed to organize Nordic cooperation on the reform of mathematics teaching. The Nordic Council set up the Nordic Committee for Modernizing Mathematics Teaching, NKMM. The committee, which was active from 1960 until 1967, appointed teams of writers. Its focus was on the mathematical content, and the teaching of seventh to twelfth grades was its main object. Joint Nordic manuscripts were successively ready until the beginning of 1966 and were translated to each language (Gjone, 1980, II. pp. 78–80; Nordisk råd, 1967).

Denmark was one of the countries which went furthest in introducing the modern New Math into university and high school programs (Karp, 2015). Iceland was a part of the Danish Realm until 1944, and in the early 1960s, Iceland was still culturally dependent of Denmark with many students receiving their training there. Iceland did not participate in the NKMM cooperation but all the Danish representatives in the committee made an impact in Iceland through their writings. G. Arnlaugsson (1966), who stayed in Denmark for education and work during 1933–36 and 1939–45, wrote his textbook *Tölur og mengi* [*Numbers and sets*] for lower secondary school level, and simultaneously presented the NKMM-material for primary level to Icelandic authorities in 1966. The New Math was introduced for experimental reasons at both school levels from 1966, and the primary level material was implemented in the majority of Reykjavík primary schools in 1967 (Bjarnadóttir, to appear).

ANALYSIS OF TEXTBOOKS

The majority of mathematics textbooks for lower secondary school level in use in Iceland during 1949–2007 were analysed in order to trace New-Math influences. By law (no. 22/1946) the lower secondary level was divided into two streams, one for prospective vocational training, and the other for college-bound education. Both had a syllabus, represented by a textbook, here named JSJ (Gissurarson and Guðmundsson, 1949–1950), with a review of fractions, proportions and percentages in the form of Rule of Three, and equations, as well as computations of area and volume. In addition, the college-bound stream had a special curriculum for those who wished to attempt an entrance examination into high schools. Its syllabus included introduction to algebra by a textbook referred to here as ÓD (Daníelsson, 1951). The two streams were united in 1976.

Syllabus changes that followed the introduction of the New Math in lower secondary schools were based on a draft of a curriculum document (Landsprófsnefnd, 1968, pp. 56–60), preparing for the introduction of the New Math to the country-wide examination. The aim was to base school mathematics on the basic concepts of the set theory, which simultaneously were simple and general and to increase emphasis on the meaning and nature of numbers and of number computations. Official curriculum documents for compulsory schools, including lower secondary level, were only published in drafts until 1989 and in a detailed edition in 1999 (Menntamálaráðuneytið, 1989; 1999). They therefore did not reflect the reform waves of the 1960s and 1970s. The process may be divided into three steps of implementation and three reaction steps.

1. The first step included the textbook *Numbers and Sets [Tölur og mengi]*, GA, by G. Arnlaugsson (1966), intended for college-bound students, written as a complementary text to the ÓD algebra textbook. The topics of GA were on one hand numbers and number relations, such as divisibility and number notation to other bases than ten. On the other hand there were sets and set algebra, and some logic. In his forewords, the author stated that the basic concepts of logic and set theory would facilitate understanding, even for small children. This textbook was ground-breaking for the New Math in Iceland, written under Danish and American influences (Bjarnadóttir, 2015).
2. The second step was translating the Nordic NKMM-material into Icelandic. From 1970, a textbook series for lower secondary level, NK (Bergendal, Hemer and Sander, 1970; Kristjánadóttir and Lárússon, 1970; Björnadóttir, 1972), was in use for a few years.
3. The third step was domestic material HL, by H. Lárússon (1972–1976), somewhat tailored after the NK but more comprehensive. The series dominated the market into the late 1980s.
4. A series by Kristjánadóttir et al. (1988–1991), SFG, less oriented towards set theory than earlier series, was published but not completed. By the time of its publication, the New Math had been presented to the majority of pupils entering lower secondary level, and therefore there was less need to introduce basic set theoretical concepts.
5. In the late 1980s, a Swedish series, ASG by Björk, Björkstén, Brodin, Ernestam, and Ljungström (1987–1989), was translated and used into the 2010s.
6. In the early 2000s, a domestic series, GP by Pálsadóttir and Gunnarsadóttir (2005–2008) was published.

Last names of the authors in the order as on the paper

The comparison of the content of the Icelandic lower secondary level textbooks to Choquet's and Servais's proposals, and the recommendations of the Royaumont seminar about arithmetic and algebra at secondary level, splits into three parts. The three areas of study are: set theoretical concepts and notation and its connection to geometric presentation in the coordinate system; structure of the number field; and the study of numbers. The focus is on the three New-Math texts of Steps 1, 2 and 3, but they are compared to two older Icelandic textbook series, JSG and ÓD, and three more recent ones, the SFG-, ASG- and GP-series.

RESULTS

In the following tables, the numbers indicate the age level when the topic in question was introduced. Age within parentheses indicates that the topic was only marginal and not worked with in what followed. Normally, the topics were readdressed regularly after they had been introduced.

Introduction of set-theoretical concepts

Textbooks	JSG	ÓD	GA	NK	HL	SFG	ASG	GP
In use	1950- 1976	1927- 1976	1966- 1976	1970- 1975	1972- 1990	1979- 1999	1987- 2015	2005- 2015
Age level	13-15	15	15	13-15	13-15	13-15	13-15	13-15
Sets			15	13	13			13
Subsets			15	13	13			
Union			15	13	15			13
Intersection			15	13	13			13
Set difference			15					
Modern symbolism			15	15	13			14
Algebraic properties of sets			15					
Cartesian product			15	15	14			
Mappings				15	14			
Functions					14			
1-to-1 correspondences			15	15	13			
Coordinate system	(14)		15	15	14	13	14	13
Coordinate geometry				15	14	13	14	13
Vectors					14			
Linear equations				15	15	13		
Logic			15					(14)
Logical symbols			15	(14)	(14)			

Table 1: Set-theoretical concepts in eight textbook series.

The textbook GA that was written for the college-bound stream for the 15-year age group as complementary to ÓD introduced all relevant set-theoretical concepts with modern symbolism, and the algebraic properties of sets, i.e. introduction to set algebra, presented in exercises, in addition to logical symbols and introduction to logic, all of which had neither appeared in JSG nor in ÓD.

The Nordic NK-series and the HL-series introduced set-theoretical concepts and notation except set difference, Cartesian product represented geometrically on the plane of $Z \times Z$, mappings and one-to-one correspondences during age 13–15, but did not explicitly address algebraic properties of sets.

The NK-series used some logical symbols but did not venture further into logic. The set-theoretical concepts and notations were not mentioned in later texts after HL, such as SFG or in the Swedish ASG, until some were again introduced in GP as aids to counting and probability.

Structure of the number field

Textbooks	JSG	ÓD	GA	NK	HL	SFG	ASG	GP
In use	1950- 1976	1927- 1976	1966- 1976	1970- 1975	1972- 1990	1979- 1999	1987- 2015	2005- 2015
Age level	13-15	15	15	13-15	13-15	13-15	13-15	13-15
The commutative laws		15		14	13	13		13
The associative laws		15		14	13	13		13
The distributive laws		15		14	13	13		13
Negative numbers	14	15		14	13	14	14	<13
Additive identity		15		14	13			13
Additive inverse		15		(14)	13	14		14
Multiplicative identity		(15)		14	13	13		13
Multiplicative inverse		(15)		14	14	14		15
0 in multiplication		15		14	13			13
0 in division		15		15	15	(14)		13

Table 2: Structure of the number field in eight textbook series

The new GA did not touch algebra as it complemented ÓD which presented explicitly all axioms of Q except multiplicative identity and inverse.

All the axioms of Q were systematically constructed in NK and HL and successively shown to apply to natural numbers, whole numbers and rational numbers.

The axioms were referred to in SFG and GP as aids to computations but rather for explaining computation and not for structural purposes. They were not at all used in the translated Swedish ASG.

The study of numbers

Textbooks	JSG	ÓD	GA	NK	HL	SFG	ASG	GP
In use	1950- 1976	1927- 1976	1966- 1976	1970- 1975	1972- 1990	1979- 1999	1987- 2015	2005- 2015
Age level	13-15	15	15	13-15	13-15	13-15	13-15	13-15
Even and odd numbers			15		13	13		13
Other classes of numbers			15		13			
Prime numbers	(14)		15	14	13	13		13
Factoring	14		15	14	13	13		13
Prime factoring	(14)		15	14	14	13		13
Squares, cubes			15			13	13	13
Divisibility			15	14	13	13	(13)	13
Series of squares, etc.			15			13		
Bases other than ten			15	14	13			
Modular arithmetic						13		
Scientific notation	(14)		15	14	14	14	14	14
Approximation				14	14	13	13	13
Square roots	14				15	14	15	15
Statistics, introduction	(14)				15	13	13	13
Probability, introduction					15	14	15	13

Table 3: Study of numbers in eight textbook series

The introduction of the New Math reinvigorated the study of numbers that had appeared in early 1900s textbooks and was only marginally touched upon in JSG. The main topics were even and odd numbers, prime numbers, squares and cubes and series of those, divisibility, bases other than ten, modular arithmetic, scientific notation, and approximation. All those topics were included in GA, and some in NK. Most of them were also included in HL, SFG and GP in addition to introduction to statistics and probability. In HL, SFG and GP, prime factoring was used to find the greatest common factor and least common multiple of two numbers. In the 2000s, most aspects of number study, introduced or revived through the New Math in the 1960s and 1970s were still there, except arithmetic in bases other than ten and modular arithmetic.

CONCLUDING REMARKS

One can read between the lines that mathematicians were enchanted by Piaget's idea that the structure of the mind aligned with the structure of modern mathematics, and Choquet proposed numerous structures to study at schools within the framework of sets and sets algebra. Servais

emphasized that the properties of the algebra of sets should be discovered rather than expounded and Arnlaugsson followed his advice, presenting them as an exercise.

Where the textbooks presented the new language and set-theoretical notation by a top-down method, it proved too laborious to learn and became a hindrance rather than facilitating the learning of mathematics at lower secondary school level. Later, they entered the textbooks in a more natural way as support concepts and not as goals in themselves.

The axioms of the number systems were widely presented and in some cases emphasized as a complete structure but no Icelandic text included group structure at lower secondary level or abstract algebra in general, nor were vector spaces. In the latest textbooks the axioms entered as support concepts to computation. Euclidean algorithm, Diophantine equations and Peano's axioms were nowhere mentioned. Presumably they were thought to better suit the upper secondary school level. Upper secondary level textbooks in use were written in foreign languages, not translated and are not discussed here.

The revived interest in the structure of numbers with prime numbers as building blocks, the increased relations between algebra and geometry through the use of coordinate geometry, unifying algebra and geometry, and introduction to new topics, such as probability and statistics, provided a new spark to school mathematics in Iceland. It broadened the mathematics spectrum in lower secondary schools for the 13–15 year age, and thus contributed to lasting influences from the New-Math reform wave.

The textbooks GA, NK and HL were written under strong influence from the Royaumont meeting and can as such be considered experimental texts since no official curriculum documents supported them. The texts SFG and ASG may be considered to be reactions to the New Math wave, implementing only what was considered viable of the reform wave at their time. The GP-series was written under a comprehensive national curriculum and thus including all topics listed there but using them as a support to a general syllabus considered suitable for 13–15 year olds.

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