RATIOS AND PROPORTIONS IN ICELAND 1716–2016

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Abstract

The topics of ratios and proportions are investigated in the oldest Icelandic arithmetic textbook, Arithmetica Islandica of 1716, preserved in the manuscript Lbs. 1694 8vo. The conjecture that it is a translation of the printed Danish Arithmetica Danica of 1649 is rejected, while there may be some transmissions of ideas. In continuation, the history of teaching ratio and proportions from Arithmetica Danica until the latest textbook in Icelandic, Skali of 2016, is investigated and related to recent research on obstacles in proportional reasoning.

Introduction

Arithmetic textbooks, written in the Icelandic language, have a history from 1716 until present. This study concerns both ends of this sequence. The content of *Arithmetica Islandica* of 1716, as preserved in a manuscript, is analysed with respect to works influencing it. Certain sources suggest that *Arithmetica Danica*, written in Latin by Geo Frommius (1649), was a model for *Arithmetica Islandica,* which in turn might be considered as an abridged version, comparing the sizes of the textbooks. *Arithmetica Islandica* may, however, have more models, and exercises in *Arithmetica Islandica* are also found in other younger arithmetic manuscripts.

The study focuses on the history of teaching ratios and proportions, and a comparison of its teaching in arithmetic textbooks. The books range from *Arithmetica Danica* and *Arithmetica Islandica* to *Skali*, an adapted translation of a Norwegian textbook series (Tofteberg, Tangen, Stedøy-Johansen & Alseth 2017) into Icelandic, the latest textbook series for the lower secondary level.

Textbook writing in Icelandic

During seven centuries from around 1100 until 1800 there were two episcopal seats in Iceland, with cathedrals and cathedral schools, one of each in Northern Iceland and in the South. The schools served to educate priests and officials. As Iceland belonged to the Danish Realm, university education had to be sought in Copenhagen, Denmark. A printing press, imported in the mid-16th century, mostly printed religious books, while secular works were printed in Copenhagen. After around year 1800, the printing press was no longer in the charge of the Church, and gradually, education literature became produced domestically.

The population of Iceland was 50,000 in 1703 and did not rise until the 19th century. The population in 2017 was 340,000. Persons, knowledgeable in Latin in the early 18th century, were 245 clergymen, 7 headmasters and teachers, and about 60 other graduates from one of the two schools and even from a university abroad, as recorded in the 1703 census (Statistics Iceland, Table 3.2, Occupations in 1703). In spite of a great distance from the mainland of Europe, which made communication by sailing only possible during summer, a considerable collection of European books existed in Iceland in possession of the learned elite, the clergy or the cathedrals. Among them were the mathematics books *Arithmeticae practicae methodus facilis* by Gemma Frisius, published in Antwerpen in 1540; *Arithmeticae libri Duo* by Petrus Ramus, Basil 1569; *Arithmetica Danica* by Jørgen From alias Geo Frommius, Copenhagen 1649; and *Compendium Arithmeticum eller vejviser* by Søren Matthisen, Copenhagen 1680 (Bjarnadóttir 2007, 58, 61–62; Ulff-Møller 2008).

Foreign literature was often translated, frequently in extracts. German and Danish books were the main foreign literary sources in Iceland after its Lutheran Reformation in 1550. The educated public, mainly clergymen, prepared the first copies of manuscripts by translating and/or adapting

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Gert Schubring, Lianghuo Fan, Victor Geraldo (eds.): *Proceedings of the Second International Conference on Mathematics Textbook Research and Development.* Rio de Janeiro: Instituto de Matemática, Universidade Federal do Rio de Janeiro, 2018.

foreign texts, or even composing own texts. The manuscripts circulated from parish to parish. At least three early 18th century arithmetic texts in manuscripts exist presently in libraries. The oldest of them is *Arithmetica Islandica* in Lbs. 1694 8vo, dated 1716 on its front page. The text itself suggests the date 1733 which may be the date of the extant manuscript.

The first substantial printed arithmetic textbook in Icelandic was published in 1780. Up to 1910s, textbooks began teaching arithmetic from scratch as schools were scarce. During 1920s–1960s, the arithmetic textbook tradition continued in two phases, each intended for primary and lower secondary education. Since the 1970s, featuring a series highly influenced by the New Math, five textbook series have been published for the lower secondary school level, all by a state textbook publishing house. The latest of them is a Norwegian arithmetic textbook series, recently translated into Icelandic, *Skali* (Tofteberg et al. 2015; 2017).

Learning Ratios and Proportions

For centuries, proportional problems were solved by a method called in Latin *Regula Trium*, the Rule of Three. The method consists of finding the fourth proportional to three known quantities. It is traced back to Italian merchants in late medieval times, described in arithmetic books, the *libri d'abbaco* (Van Egmond 1980), while the way of thinking in the Rule of Three can be found in ancient Indian works by Brahmagupta (597–668) and Bhaskara II (1114–1185) (Tropfke 1980).

Proportional reasoning is considered a unifying theme in mathematics. It involves a sense of co-variation and the ability to make multiple comparisons in relative terms. The skills needed for proportional reasoning include multiplicative and relational thinking; and a highly developed understanding of foundational concepts, including fractions, decimals, multiplication, division, and scaling (Van de Walle, Karp & Bay-Williams 2010).

Many researchers have elaborated on students' understanding of proportions and proportional concepts. Keranto (1994) lead a teaching experiment focussing on developing proportional reasoning and ratio concept in the eighth grade where the problems were first learned to be solved mentally, emphasizing the unit-rate method, then in writing, using proportions. In this way, the pupils' real-world experiences and spontaneous models of solution were utilized naturally in teaching.

De Bock, Van Dooren and Verschaffel (2013) conducted two studies on students' ability to model textual description of situations with different kinds of representations of functions:

- proportional, $y = \mathrm{k}x$
- inverse proportional, $y = \mathrm{k}/x$, $x \neq 0$, and
- affine, $y = \mathrm{k}x + \mathrm{b}$, $\mathrm{b} \neq 0$.

Their results indicate that

- students tend to confuse these models, and
- the representational mode has an impact on this confusion.

When investigating students' ability to link representations of proportional, inverse proportional, and affine functions to other representations of the same functions, results indicated that students make most errors for decreasing functions. The number and nature of the errors also strongly depended on the kind of representational connection to be made. In both studies, mutual confusion between two increasing, and between two decreasing functions was reported. Both studies provided evidence for a strong impact of representations in students' thinking about these different types of functions. In a mathematical modelling context, graphical representations were helpful in most cases to detect the model underlying a realistic situation. For mutually connecting representations, tabular representations, providing concrete function values, proved to be most supportive.

The study

The following study is divided into two parts:

1. Comparison of *Arithmetica Danica* (Frommius 1649) and *Arithmetica Islandica*, contained in manuscript Lbs. 1694 8vo (1716/1733).

2. Presentations of ratio and proportions in 18th, 19th and 20th century books, up to *Skali* (Tofteberg, et al.2015; 2017),

The questions are:

1. Is Arithmetica Islandica an adapted extract of Arithmetica Danica?
2. How has the presentation of ratio and proportions developed during the 300-year period between the two arithmetic works, *Arithmetica Islandica* and *Skali*?

The contents of *Arithmetica Danica* and *Arithmetica Islandica* will be listed side by side, thus comparing their order and length of content. The *Regula Trium Directa* and *Regula Trium Inversa*, the ancient rules to solve directly proportional and inversely proportional tasks, will be examined and contrasted by other methods, such as verbal, tabular, algebraic, geometrical and graphical representations.

The two arithmeticas

Arithmetica Islandica

The manuscript *Arithmetica Islandica* of 1716 is the oldest arithmetic textbook in the Icelandic language from modern times. Earlier works are encyclopaedic. This arithmetic treatise on 73 handwritten sheets, 146 pages, is a part of a larger manuscript, Lbs. 1694 8vo, contained in pages 37r–109v. A senior enforcement officer of the 18th century Iceland, Skúli Magnússon (1947), recounts in his biography that his father, clergyman Magnús Einarsson (1675–1728), had made a free translation of *Arithmetica Danica*. The time period and name of the treatise suggests that *Arithmetica Islandica* could be its free and adapted translation.

Figure 1: The title: "*Arithmetica Islandica* Skrifað Anno MDCCXVI" (*Lbs. 1694 8vo*, 37r).

On its front page, it says "*Arithmetica Islandica*, Skrifað Anno MDCCXVI" [… written in 1716]. Later in the work, the year 1733 is mentioned as the current year's datum, suggesting that the date of the extant copy is 1733.

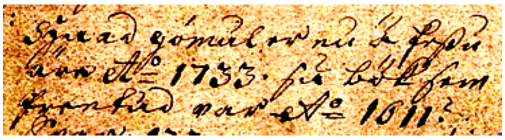

Figure 2. "How old is now in this year, A° 1733, the book which was printed in A° 1611?" (*Lbs. 1694 8vo*, 54v).

After discussing the meaning of the terms *arithmetica*, *geometrica*, *astronomia* etc., the text continues into *Numeratio*, numeration, reading large numbers in Hindu-Arabic number notation as well as Roman notation, similar to *Arithmetica Danica* without any identical examples. The text goes further into monetary, measuring and time units. These topics were important for the Icelandic public in their trade with foreign merchants while these matters are not mentioned in *Arithmetica Danica*. The treatise continues through the four arithmetic operations in whole numbers and the various units. Both works explain *Probatio*, that multiplication and division can probate each other.

The second section, *Progressio*, is about sequences, such as the odd numbers: 1, 3, 5, 7, …; and every third number: 1, 4, 7, 10, …; the even numbers: 2, 4, 6, 8, 10, … and then further into decreasing sequences, followed by geometric sequences: 1, 2, 4, 8, 16, …; 1, 3, 9, 27, … The third section concerns fractions; first a definition, then reduction, and the four arithmetic operations. The fourth section is called *Regula Trium*, the Rule of Three, which is claimed to be most useful and indispensable to all those who exercise the art of reckoning.

Arithmetica Danica

The title of the book is *Arithmetica Danica seu brevis ac perspicua institutio arithmeticae vulgaris* (Frommius 1649). It is a total of 164 pages, written in Latin by the Dane Jørgen From whose Latinised name is Geo Frommius. Book one is named *De Arithmetica Simplici*, on simple arithmetic: numeration, including Roman numerals, and the four arithmetic operations in whole numbers; notation of fractions, their reduction and least common multiple, and the four operations; and extraction of quadratic and cubic roots.

Book two is named *De Arithmetica Comparata*, on comparative arithmetic. It contains ratios between numbers, proportions and progressions, i.e. sequences; *Regula Trium*, i.e. Rule of Three; the inverse Rule of Three, the composite Rule of Three, *Regula Societatum* and *Regula Falsi* together with further elaboration on proportions.

Arithmetica Danica was a registered property of the South Cathedral Skálholt in 1744 (Ágústsson & Eldjárn 1992). Concerning the conjecture that clergyman Magnús Einarsson wrote the *Arithmetica Islandica* one must consider that a clergyman in Northern Iceland may only have seen *Arithmetica Danica* but not had it by hand. Einarsson, however, had a young mathematically-inclined teacher, Jón Árnason, at the North Cathedral School, Hólar, from 1692. Árnason became bishop and served at the south episcopal seat Skálholt during 1722–1743 (Ólason 1950). The estate at his death, dated May 4, 1743, reveals that he possessed foreign mathematical books, among them *Arithmetica Danica*, the aforementioned Sören Matthisen's *Arithmetica*, and a biblical *Arithmetica* by Jacob Borrebye (The National Archives of Iceland). This leads to the conjecture that Árnason brought the book with him from his studies in Copenhagen in 1722 when he began teaching at the Hólar North Cathedral School. Einarsson, then 17-year-old, may have made his own copy from his teacher's notes, tailored after *Arithmetica Danica*. Bishop Árnason may have bequeathed the book to Skálholt cathedral later.

We know that Bishop Árnason was the best mathematician in Iceland of his time, but indeed there was not much competition. He published a book in 1739, *Dactylismus Ecclesiasticus eður Fingra-Rím,* (Árnason 1838), to present the Gregorian Calendar that was adopted in year 1700 in the Danish Realm. It also introduced for the first time in print the domestic farmers' calendar, an ancient week-based calendar (Bjarnadóttir 2016).

Comparison of contents of Arithmetica Danica and Arithmetica Islandica

In *Table 1*, the length, size and overview of the contents of *Arithmetica Danica* and *Arithmetica Islandica*, are presented. The reader should keep in mind that *Arithmetica Islandica* was a manuscript in a smaller size than the printed *Arithmetica Danica*.

Arithmetica Danica (108 pp. in 4°)		Arithmetica Islandica (146 pp. in 8°)	
	Book 1, De Arithmetica Simplici	[I]	
Cap. 1–2 (17. p.)	Definitions, number notation, literature examples	Introduction (19 pp.)	Translation of terms, number notation, measuring and monetary units
Cap. 3–6 (20 pp.)	Addition, subtraction, multiplication, division	(56 pp.)	Addition, subtraction, multiplication, division
Cap. 7–13 (15 pp.)	Fractions: notation, reduction, the four operations, compound fractions		

Cap. 14–15 (13 pp.)	Extraction of square root and cubic root		
	Book 2, De Arithmetica Comparata		
Cap. 1–2 (11 pp.)	Ratios, proportions and progressions	II Progressio (14 pp.)	Progressions
		III Fractions (25 pp.)	Fractions: notation, reduction, the four operations
Cap. 3–4 (9 pp.)	Regula Trium Directa, Regula Trium Inversa	IV Regula Trium (16 pp.)	Regula Trium Directa, Regula Trium Inversa aut Obliqua
Cap. 5–7 (19 pp.)	Regula Trium Composita, Regula Societatum, Regula falsi	(16 pp.)	Regula Trium Dupla, Regula Duple Reciproca, Regula Alligationes, Regula Consortio

Table 1: Comparison of the contents of Arithmetica Danica and Arithmetica Islandica

One notices that the order of the contents is different. Progressions are presented before fractions in *Arithmetica Islandica*. Furthermore, the examples given are completely different. The Icelandic examples refer to Icelandic environment and circumstances, while in *Arithmetica Danica* many of them are historical.

Comparing Regula Trium Directa

Arithmetica Danica

The *Regula Trium*, the Rule of Three, is introduced by the following text (in a crude translation):

> On account of its immeasurable usefulness, this proportional rule deservedly is highly valued as Aurea [Golden]: which, given that out of three known parts, with definite calculation of arranged numbers, leads the way to the fourth (Frommius 1649, 76).

In continuation, the author refers to Euclid's *Elements*, proposition 19, book 7:

> If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and the third; and if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional (Euclid 1956, 318).

The author takes an example of that 2 times 9 is equal to 3 times 6:

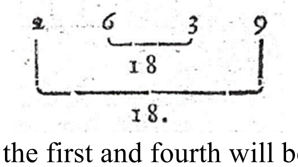

Figure 3: "… the number produced from the first and fourth will be equal to the number produced from the second and the third …", (Frommius 1649, 77).

The author then describes the method of the Rule of Three: Four numbers are to be arranged so that the first and the third are of the same kind and the second of the same kind as the fourth. When one wanted to know the fourth number, the second and the third should be multiplied together. The product should be divided by the first number to give the sought after fourth number (Frommius 1649, 77–78). Example:

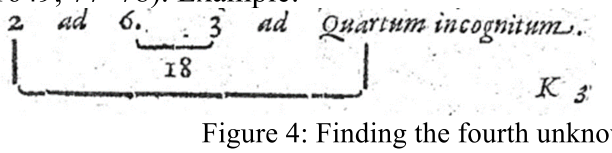

Figure 4: Finding the fourth unknown proportional (Frommius 1649, 77).

Arithmetica Islandica
The introduction to the direct Rule of Three tells the reader that the rule is indispensable for all those who practice the art of reckoning and can rightly be called [*Regula*] *Aurea*, that is the golden rule, because as gold stands out of other metals, so it surpasses other rules. The method presented is similar to that in *Arithmetica Danica*: The first and the third numbers are to be of the same kind, and the second of the same kind as the unknown fourth. The second and third numbers are to be multiplied together and divided by the first to gain the fourth (Lbs. 1694 8vo, 94r–95r). The problems are domestic. A typical question is: If 30 eiderduck-eggs weigh 12 marks, what then weigh 135 [eggs]?

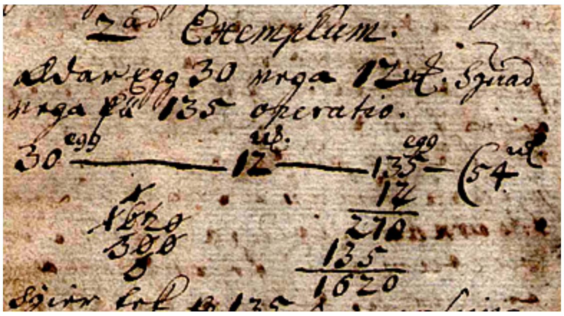

Figure 5: The weight of 135 eiderduck-eggs by the Rule of Three (Lbs. 1694 8vo, 95v).

Comparing Regula Trium Inversa

Both books explain the inverse rule such that the less the first number is to the third, the more is the second to the fourth unknown, and vice versa, assuming the same sequence as before.
An example from Arithmetica Islandica:

> To complete a work during 16 weeks, 9 men are needed. How many men are needed for the same work during 24 weeks? (Lbs. 1694 8vo, 100v).

Solution: Multiply the first and second numbers and divide the product by the third to gain 6 men, see Figure 6.

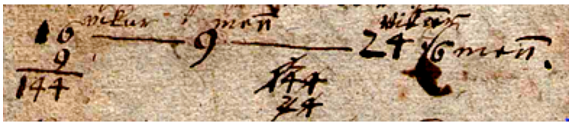

Figure 6: Finding the number of men to complete a work by the inverse Rule of Three (Lbs.1694 8vo, 100v).

Examples in Arithmetica Islandica also found in other works

Several arithmetic examples have circulated in Protestant Europe. A textbook by the Protestant Sigismund Suevus (1593), *Arithmetica Historica. Die löbliche Rechenkunst*, contained a number of arithmetic examples, disguised in biblical dress. These examples showed up in later textbooks, such as Euler's (1738) *Einleitung zur Rechenkunst*, and at least two Icelandic textbooks in manuscripts, one of them *Arithmetica Islandica* and the other manuscript *ÍB 217 4to Arithmetica – Það er reikningslist* [*That is Reckoning Art*], presumably written in 1721 (Bjarnadóttir, 2011).
Among those examples is a story about the age of Methusalem. Examples about the number of hours in a year, and the circumference of the Earth, both appear by Suevus (1593), *ÍB217 4to* of 1721, and Euler (1738), illustrating how examples were copied from one textbook to another in early modern times. Neither of these examples, however, appear in *Arithmetica Danica*, which is entirely void of practical examples, nor in *Compendium Arithmeticum* by Matthiesen (1680) which is known to have been in the possession of Bishop Árnason (The National Archives of Iceland).

Presentations of ratio and proportions in recent centuries

Iceland in 1780–1900

A number of calamities fell upon Icelanders in the 18th century. Among Danish subsidies after mid-18th century, were grants to printing the first substantial arithmetic textbooks, written in Icelandic by Olavius (1780) and Stefánsson (1785), both printed in Copenhagen. No further mathematics textbooks were printed until 1841. Olavius (1780, 172–178, 290–294) explained that the task in the Rule of Three, the Golden Rule (*regula aurea*), was to find the fourth term in a geometric equality. The numbers were to be arranged with the fourth term missing, such as 4 – 12 – 6 – *X*. Then the working method was: Multiply *middle term* and *rear term* and divide by the *front term*.

Stefánsson's (1785, 132–137) method to find the fourth proportional number is the same as in *Arithmetica Danica*: multiply the second and the third term and divide by the first term, assuming the same order. Stefánsson's son edited the book when preparing it for print in Copenhagen. In that new version, the phrase is found: "it is also called the golden rule or *Regula aurea* for its supremacy, because as much superiority as gold has over other metals, so much it surpasses other arithmetic rules" (Stefánsson, 1785, 132). The similarity of this phrase to that of *Arithmetica Islandica*, especially the word "yfirgengur" [surpasses] suggests that the son knew that work. However, the term *Regula aurea* may be found in many, or most, arithmetic textbooks of early modern age.

Briem (1869) wrote an influential textbook, used in the emerging lower secondary schools from the 1870s. The search for the unknown in direct Rule of Three was to find a number that is as many times greater or less than the *middle term*, as the *rear term* is greater or less than the *front term* in the sequence of the three known proportional numbers.

England in the 19th century

J. Stedall (2012, 56) said in The history of mathematics – A very short introduction:

> The Rule of Three was a rule that enabled countless generations of students to answer questions like: A men dig a ditch in B days, how long would it take C men to do the same job?
>
> A 19th century English century school boy was not expected to start doing anything on his own initiative. He would be taught that he must multiply A by B and divide by C.

Problems of that kind, digging ditches, were still taught in Iceland in the 1970s (Gíslason 1962, 52).

Iceland in 1920–1960

Mathematician Ó. Daníelsson was an undisputed leader of secondary mathematics education during 1920s–1960s. He presented by examples two cases, the direct and the inverse Rule of Three:

Case I:	Case II:
4 meters cost 3 crowns	4 men complete a work in 3 days
6 meters cost *x* crowns	6 men complete the work in *x* days

In the two cases the student is to consider the ratios 6/4 and 4/6. Which to choose depends on whether the outcome should be greater or less than the known number 3.

In case I, one should choose to multiply 3 by 6/4 to gain the answer 4 ½ crowns.

In case II, one is to multiply 3 by 4/6 and the answer is 2 days (Daníelsson 1938, 45–46).

Gíslason (1962, 45–52) tried to modify the method by inserting a unit sentence:

1 meter costs ¾ crowns	1 man completes the work in $3 \cdot 4$ days

This procedure was expected to make it easier for the reader to decide if to continue by multiplying or dividing.

Present times

Skali 2015–2017

By the introduction of the New Math, methods and procedures became objects of scrutiny. The term Rule of Three disappeared. Proportions, however, continued in the curriculum. With improved printing technology, they became represented in a great variety of ways: verbally, tabular, geometrically, graphically, and algebraically in functions. The latest arithmetic textbook series for the lower secondary school level is a Norwegian series in six volumes, *Maximum*, termed *Skali* in Icelandic. In *Skali 2A* (Tofteberg et al. 2015) for the 14-year age, the ratio of mass to volume is examined by the students by drawing a double number line, see Figure 7.

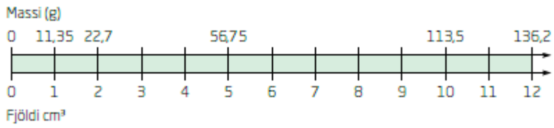

Figure 7: The ratio of mass in kg to volume in cm^3 (Tofteberg et al. 2015, 172).

Students are also trained in recognizing direct proportionality as a linear function including the pair of coordinates of Origo, (0,0). In continuation, in *Skali 3B* (Tofteberg et al. 2017), they learn about the characteristics of the graph of inversely proportional quantities, see Figure 8.

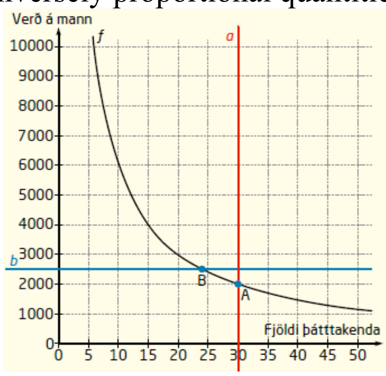

Figure 8: Sharing cost between varying number of participants (Tofteberg et al. 2017, 30).

Students also practice distinguishing between the various types of functions, such as:

- proportional (4): $y = kx$
- inversely proportional (3): $y = k/x$,
- quadratic (1): $y = k \cdot x^2$
- affine (2): $y = kx + b$, see Figure 9.

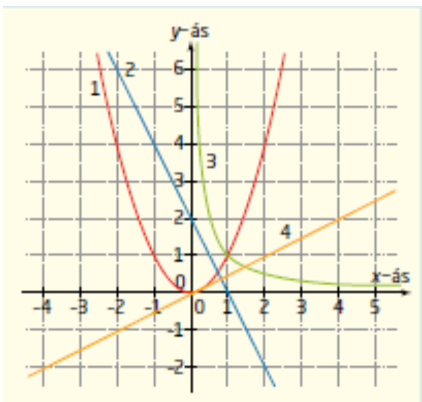

Figure 9: Students practice distinguishing between different types of functions (Tofteberg et al. 2017, 28).

They also practice reading tables, and decide whether the relation between two variables, x and y, adheres to $y/x = k$, $(x \neq 0)$ or to $x \cdot y = k$, see Figures 10 and 11:

a

x	3	4	8	12
y	9	12	24	36

b

x	0	5	7	15
y	0	10	21	60

Figure 10: Exercises in reading tables (Tofteberg et. al. 2017, 28).

x	3	10	12
y	20	6	5
y · x	60	60	60

Figure 11: Quantities in inverse proportions (Tofteberg et al. 2017, 41).

Discussion

The question if *Arithmetica Islandica* was a translation or an adapted extract of *Arithmetica Danica* can answered negatively. We see that the structures of the works are similar, but both of them also adhere to a pattern of arithmetic textbooks that had been developed since the early modern age. No examples are the same, not one, and even the phrases about the usefulness and superiority of the Rule of Three, the Golden Rule, are general and can be found in so many books that they cannot be claimed to be related. However, the author of *Arithmetica Islandica*, who most likely was a scholar that graduated from one of the two cathedral schools, must have had access to foreign books and a knowledgeable teacher, who actually was available in the northern Hólar Cathedral School. We may conjecture that he knew *Arithmetica Danica* and wanted Icelanders to have a similar work.

Neither can we confirm that the Reverend Magnús Einarsson (1675–1728) wrote *Arithmetica Islandica*. But having been a student of the young scholar Jón Árnason, later Bishop Árnason (1665–1743), makes it quite likely that he had acquired knowledge to complete such a work. In 1716, Einarsson was in his early forties and could have had time to collect examples from his experience and from books that he may have seen or possessed. In the thinly populated country, there are not many candidates for such an enterprise. Another candidate is the Bishop himself. The facts that weigh against him are that his works are well documented and available in archives. Besides he had more possibilities than others to have his works printed, at least in Copenhagen. The third candidate is named Magnús Arason Thorkelin (–1728), titled a sea captain. He served with the mathematician Ole Römer in Copenhagen and later in the Danish army on geodesy. He lived abroad for 16 years, spoke a number of European languages, and returned to Iceland first in 1721 (Ólason 1950), after the assumed date of the original manuscript of *Arithmetica Islandica* of 1716. He may, however, rather have been the author of *Arithmetica – That is reckoning Art* of 1721, contained in manuscript ÍB 217 4to, which does not refer to Icelandic environment (Bjarnadóttir 2011). Einarsson is therefore the most promising known candidate for authoring *Arithmetica Islandica*.

The Rule of Three, the Golden Rule, has been useful through the centuries, especially in trade. Proportional reasoning is still important, for example in the task of converting from one currency to another. The methods practiced are more debatable. The old versions of arguments supporting methods for solving direct proportionality or inverse proportionality, can also apply to other kinds of functions, such as quadratic functions or affine functions and are therefore insufficient and confusing. The unit-rate method, used by Gíslason (1962) and emphasized by Keranto (1994) was an effort to improve the Rule of Three and is still in use where applicable.

The context is also important. Many generations have planned works such as digging ditches even if few workers and still fewer teenagers are occupied with that kind of work anymore. The contexts in examples on proportionality have only lately approached the environment of contemporary youth, such as enlarging pictures, or planning activities and sharing cost. New topics, such as statistics and probability, have brought opportunities and needs for applying proportional reasoning.

Presently, other kinds of functions than proportionality appear frequently. The difficulty is to decide if proportional reasoning is applicable or not. Referring to the research of De Bock et al. (2015), students tend to confuse situations with different kinds of representations of proportional, inverse proportional, and affine functions, and the representational mode has an impact on this confusion. In a mathematical modelling context, graphical representations were helpful in most cases to detect the model underlying a realistic situation, while for mutually connecting representations, tabular representations providing concrete function values, proved to be most supportive.
We see that *Skali*, as a representative for contemporary textbooks for teenagers, has adhered to trends that are discussed by De Bock et al. (2015) in considering that representation is important, such as practice in tabular representation, and practice in distinguishing between the various types of increasing and decreasing functions. *Skali* also offers a variety of contexts that concern contemporary teenagers. The presently available versatile representations – verbal, tabular, algebraic, geometrical and graphical – hopefully will serve as aids in reducing the number of pitfalls meeting students.

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