On the Uniqueness of the Solution to a Variational Problem in Image Processing

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1. Introduction

Variational models for separating an image into a *cartoon* component and a *noise* or *texture* component have been widely studied. The best known of these is the Rudin–Osher–Fatemi model which for a given image $f \in L^2$ asks for the minimizer of

$$E_{ROF,K}(u) = \|u\|_{BV} + \lambda \|f-u\|_2^2.$$

Here u will be the cartoon component, $f-u$ the noise or texture and $\lambda > 0$ is a control parameter. Also note that $\|u\|_{BV}$ denotes the BV *bounded variational* norm of u, see e.g. [1].

Since the functional $E_{ROF,K}$ is strictly convex in u this model has unique minimizers. However, it is not without its problems. One problem is that even though we start with an f which we would like to think has no noise or texture, such as the characteristic function of a unit disk in $\mathbb{R}^2$ we get that the optimal solution u^* does not satisfy $u^* = f$. To alleviate this we might consider replacing the L^2 norm with an L^1 norm. In this case, provided the control parameter is large enough, the characteristic function of the disk is its own optimizer. On the other hand we have sacrificed uniqueness, in particular there exists a radius R such that if we take f to be the characteristic function of a disk of radius R then both f and the zero function are optimizers.

For a further discussion of these and other related models see [3] and references therein.

In response to a question of John B. Garnett [2] we will in this note study the model which seeks to minimize

$$E_K(u) = E_K(u,f) = \|u\|_{\mathrm{BV}} + \lambda \|K*(f-u)\|_1$$

where K is a positive, even and real analytic kernel with $\int_{-\infty}^{\infty} K dx = 1$. Eventually we will fix K to be $K(x) = \sqrt{\delta} e^{-\delta \pi x^2}$. Furthermore we will consider a periodic variant of this model where f and u are periodic of period 1 and these norms are to be calculated over an interval of length 1.

The functional we wish to minimize is convex in u but not strictly convex. Therefore the minimizer need not be unique and indeed our main result is that the minimizer is not necessarily unique.

2. Main argument

Let f be periodic with period 1 and such that

$$f = \begin{cases} -1 & -\frac{1}{2} < x \leq 0 \\ 1 & 0 < x \leq \frac{1}{2}. \end{cases}$$

When the kernel K is applied to a periodic function f we can calculate

$$\begin{aligned} K*f(x) &= \int_{-\infty}^{\infty} K(x-y) f(y) dy = \sum_{n=-\infty}^{\infty} \int_{p+n}^{p+(n+1)} K(x-y) f(y) dy \\ &= \int_p^{p+1} \sum_{n=-\infty}^{\infty} K((x-y)-n) f(y) dy = \int_p^{p+1} \tilde{K}(x-y) f(y) dy. \end{aligned}$$

Here $\tilde{K}(x) = \sum_n K(x-n)$ is periodic and p is any real number so that the integration is over one period. We also require that $\tilde{K}$ is a decreasing function of x in the interval $x \in [1, 1/2]$. (We should choose K to ensure this.)

Let $f_r(x) = -f(-x)$ be the rotation of (the graph of) f about the origin by π and similarly u_r for u. It is clear from the definition of f that $f_r \equiv f$ and it is also clear from the definition of the E-function that $E_K(u,f) = E_K(u_r, f_r)$. Hence, if u is a minimizer for $E_K(\cdot, f)$ for some fixed value of λ then u_r is also a minimizer for $E_K(\cdot, f)$. Since $E_K(u)$ is convex as a functional of u then $v = (u+u_r)/2$ is also a minimizer and v is an odd function. In what follows we will only look for these odd minimizers.

Note that

$$\|u\|_{\mathrm{BV}} \geq 2(\max_{x\in[0,1]} u(x) - \min_{x\in[0,1]} u(x)).$$

This is the only estimate we need on the BV norm and it holds in general due to the definition of the norm. Since (we are assuming) u is odd we see that if $a = \max_{x\in[0,1]} u(x)$ then $-a =$

$\min_{x\in[0,1]} u(x)$. Also $a \leq 1$ as we note that $\|f\|_{\mathrm{BV}} = 4$ and if $a > 1$ then $\|u\|_{\mathrm{BV}} \geq 4$ and f itself perfoms better than u so u is not a minimizer.

We will now manipulate the expression of the functional to suit our needs. In the following calculations we integrate over the period $[-1/2, 1/2]$.

$$\begin{aligned} E_K(u) &= \|u\|_{\mathrm{BV}} + \lambda\|K * (f-u)\|_1 \\ &= \|u\|_{\mathrm{BV}} + \lambda \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{K}(x-y)[f(y)-u(y)]dy \right| dx \\ &= \|u\|_{\mathrm{BV}} + \lambda \left(\int_{-\frac{1}{2}}^{0} + \int_{0}^{\frac{1}{2}} \right) \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{K}(x-y)[f(y)-u(y)]dy \right| dx. \end{aligned}$$

We want to know how to manipulate the absolute value in the last integral. For $0 \leq x \leq \frac{1}{2}$ let us look at

$$\begin{aligned} &\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{K}(x-y)[f(y)-u(y)]dy \\ =& \int_{-\frac{1}{2}}^{0} + \int_{0}^{\frac{1}{2}} \tilde{K}(x-y)[f(y)-u(y)]dy \\ =& \int_{0}^{\frac{1}{2}} \left(\tilde{K}(x-y) - \tilde{K}(x+y) \right) [f(y)-u(y)]dy. \end{aligned}$$

The last equality uses the fact that f and u are odd functions. As $f(x) \geq u(x)$ for $x \in [0, \frac{1}{2}]$ we only need to consider the sign of the $\tilde{K}$-prefactor. We see that for $x, y \in [0, \frac{1}{2}]$ we have $|x-y| \leq |x+y|$. We need to consider two cases:

(1) $|x+y| \leq \frac{1}{2}$: Then both $|x-y|$ and $|x+y|$ lie in the interval $[0, \frac{1}{2}]$ and we have assumed that $\tilde{K}(\xi) < \tilde{K}(\eta)$ if $|\eta| < |\xi| \leq \frac{1}{2}$. Thus $\tilde{K}(x-y) - \tilde{K}(x+y)$ is positive.
(2) $\frac{1}{2} < |x+y| \leq 1$: We note that $\tilde{K}(x+y) = \tilde{K}(1-(x+y))$ because $\tilde{K}$ is even and periodic. Furthermore we have $|x-y| \leq \frac{1}{2}$ and $|1-(x+y)| \leq \frac{1}{2}$. Moreover, we see that $|x-y| \leq |1-(x+y)|$ since $x, y \leq \frac{1}{2}$. Thus $\tilde{K}(x-y) - \tilde{K}(x+y)$ is again positive.

Finally, considering the symmetry we get the same from the integral for $x \in [-\frac{1}{2}, 0]$ as from the integral for $x \in [0, \frac{1}{2}]$. Thus

$$E_K(u) = \|u\|_{\mathrm{BV}} + 2\lambda \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\tilde{K}(x-y) - \tilde{K}(x+y) \right) [f(y)-u(y)]dy$$

For $0 \leq y \leq \frac{1}{2}$ we have that $f(y) = 1$. For $a = \max_{y\in[0,\frac{1}{2}]} u(y)$ we let

$$v(y) = \begin{cases} -a & -\frac{1}{2} < y \leq 0 \\ a & 0 < y \leq \frac{1}{2} \end{cases}.$$

Then we see that for

$$E_K(u) = \|u\|_{\mathrm{BV}} + 2\lambda \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\tilde{K}(x-y) - \tilde{K}(x+y) \right) [1-u(y)]dy$$

we have $E_K(v) \leq E_K(u)$ with equality only when $u = v$. Hence we conclude that all odd minimizers of $E_K(u)$ are of the same form as v. Let us consider $u = v$ in the last equation.

$$\begin{aligned}
E_K(v) &= \|v\|_{\mathrm{BV}} + 2\lambda \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\tilde{K}(x-y) - \tilde{K}(x+y)\right) [1 - v(y)] dy \\
&= 4a + 2(1-a)\lambda \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\tilde{K}(x-y) - \tilde{K}(x+y)\right) dy \\
&= 2 + a \left(4 - 2\lambda \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\tilde{K}(x-y) - \tilde{K}(x+y)\right) dy\right) \\
&= 2 + a\left(4 - 2\lambda\Delta(K)\right)
\end{aligned}$$

where

$$\Delta(K) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\tilde{K}(x-y) - \tilde{K}(x+y)\right) dy.$$

is a number which depends only on K.

Let us then return to the case of $K(x) = \sqrt{\delta}e^{-\delta\pi x^2}$. Then $\Delta(K) = \Delta(\delta)$ depends only on the parameter δ. We look at two limits for δ:

- In the limit $\delta \to 0$ then $\tilde{K}(x) = 1$ and hence $\Delta(0) = 0$.
- In the limit $\delta \to \infty$ then $\tilde{K}(x) = \delta_0(x)$ where δ_0 is the Dirac-delta function. Hence $\Delta(\infty) = \frac{1}{2}$.

For $0 < \delta < \infty$ we get $0 < \Delta(\delta) < \frac{1}{2}$. Then the value a for the optimal v in $E_K(v)$ is given by

$$a = \begin{cases} 0, & \lambda < \frac{2}{\Delta} \\ 1, & \lambda > \frac{2}{\Delta} \\ \text{any value between 0 and 1}, & \lambda = \frac{2}{\Delta} \end{cases}$$

That is for $\lambda < \frac{2}{\Delta}$ the constant function zero is a minimizer, and when $\lambda > \frac{2}{\Delta}$ the function f is its own minimizer. However when $\lambda = \frac{2}{\Delta}$ the minimizer is not unique.

From this we have found the condition that is needed for f to be its own minimizer. We also see that neither u nor $\|u\|_{\mathrm{BV}}$ needs to be unique.

References

[1] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[2] John B. Garnett. Personal communication, 2008.

[3] Luminita A. Vese John B. Garnett, Triet M. Le. Some variational problems arising in image processing. preprint, 2010.

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