GEOMETRIC BRASCAMP–LIEB HAS THE OPTIMAL BEST CONSTANT

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Abstract. We study the optimal best constant for the Brascamp–Lieb inequality and show that it is furnished exactly by the geometric Brascamp–Lieb inequality.

1. Introduction

The Brascamp–Lieb inequality unifies and generalises several of the most central inequalities in analysis, among others the inequalities of Hölder, Young and Loomis–Whitney. It has the form

$$\int_H \prod_{j=1}^m f_j^{p_j}(B_j x)\,\mathrm{d}x \le C \prod_{j=1}^m \left(\int_{H_j} f_j \right)^{p_j} \tag{1}$$

where H and H_j are finite dimensional Hilbert spaces of dimensions n and n_j respectively, $B_j : H \to H_j$ are linear maps, p_j are non-negative numbers, C is a finite constant and f_j are non-negative functions. We shall refer to $((B_j), (p_j))$ as the Brascamp–Lieb datum for this inequality.

The inequality goes back to Brascamp and Lieb in [4] and later Lieb [7] proved the fundamental result that gaussians exhaust the inequality in the sense that the smallest constant $BL((B_j), (p_j))$ for the Brascamp–Lieb inequality with datum $((B_j), (p_j))$ can be calculated by testing the inequality on tuples of centred gaussians.

Let (A_j) be a tuple of positive definite matrices on H_j. Then the functions $f_j(x) = e^{-\pi\langle A_j x, x\rangle}$ are centred gaussians and the smallest number C so that (1) holds for (f_j) is

$$BL((B_j), (p_j), (A_j)) := \left(\frac{\prod_{j=1}^m (\det A_j)^{p_j}}{\det(\sum_{j=1}^m p_j B_j^* A_j B_j)} \right)^{\frac{1}{2}} \tag{2}$$

as follows from a straightforward calculation, using the identity $\int_{\mathbb{R}^n} e^{-\pi\langle Ax,x\rangle} dx = (\det A)^{-\frac{1}{2}}$. The content of Lieb's theorem is that

$$BL((B_j), (p_j)) = \sup_{(A_j)} BL((B_j), (p_j), (A_j)) \tag{3}$$

where the supremum is taken over all tuples (A_j) of positive definite transformations $A_j : H_j \to H_j$.

So, $BL((B_j), (p_j))$ is the best constant for inequality (1) in the sense that the inequality holds for any $C \ge BL((B_j), (p_j))$ but fails for any C smaller than it. In this note we address the question, what is the optimal, i.e. smallest value $BL((B_j), (p_j))$ can take? Unqualified, the answer is obviously zero since the scaling $B_j \mapsto \lambda B_j$ only affects the left hand side but leaves the right hand side unchanged. Thus, it is natural to normalise B_j. Our main result is the following theorem.

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Theorem 1. *For any tuple (p_j) of positive real numbers and any linear transformations $B_j : H \to H_j$ normalised so that $B_j B_j^* = \mathrm{Id}_{H_j}$ we have*

$$BL((B_j),(p_j)) \geq 1. \tag{4}$$

Remark 2. An analysis of the optimal best constant for the Brascamp–Lieb inequality was undertaken in Section 6 of [6]. This analysis is based on equation (3.8) of that paper which unfortunately is not correct, as noted in [5]. We will take a different approach to the problem.

Remark 3. Note that any linear transformation $\tilde{B}_j : H \to H_j$ can be factored $\tilde{B}_j = E_j B_j$ where B_j is normalised as in the theorem and $E_j : H_j \to H_j$ is a linear transformation on H_j. Thus any Brascamp–Lieb inequality can be converted into one for which the theorem applies. Also, each E_j must be invertible in order for $BL((\tilde{B}_j),(p_j)) < \infty$.

Remark 4. Ball [1] and Barthe [2] have introduced geometric Brascamp–Lieb data, which in addition to being normalised according to our definition, also have the property that

$$\sum_{j=1}^{m} p_j B_j^* B_j = \mathrm{Id}_H .$$

They have shown that $BL((B_j),(p_j)) = 1$ for a geometric datum and thus Theorem 1 establishes that no normalised datum furnishes a better best constant than the geometric datum. Moreover we will prove the following theorem.

Theorem 5. *There is equality in* (4) *if and only if* $((B_j),(p_j))$ *is a geometric datum.*

Remark 6. Note that we are considering a minimax problem here, our result is that

$$\inf_{(B_j)} \sup_{(A_j)} BL((B_j),(p_j),(A_j)) \geq 1$$

where (B_j) and (A_j) come from suitable classes as described above.

2. Proofs of the theorems

We will use induction on the dimension of the space H. In case $\dim H = 1$ then the restriction $B_j B_j^* = \mathrm{Id}_{H_j}$ forces B_j to be an isometry between H and H_j, which means that we may as well assume that $H = H_j$ and B_j is the identity operator. From this we have that $BL((B_j),(p_j)) = \infty$ unless the scaling condition $\sum p_j = 1$ holds and in the latter case we are dealing with the multilinear Hölder inequality which is geometric and for which it is well known that $BL((B_j),(p_j)) = 1$.

Let us therefore assume that $\dim H > 1$ and that we have proved the results for any datum for which the dimension of the base space is smaller than $\dim H$.

Let us fix (p_j). With our normalising condition, the tuples (B_j) range through a compact set E which we can divide into three parts E_1, E_2 and E_3 from the analysis of [3]. In E_1 we have that $BL((B_j),(p_j)) = \infty$, that is the Brascamp–Lieb inequality does not hold. In E_2 the datum has a critical subspace which is defined in [3] as a proper subspace V of H such that

$$\dim V = \sum_{j=1}^{m} p_j \dim B_j V.$$

In this case Lemma 4.8 from [3] states that

$$BL((B_j),(p_j)) = BL((B_{V,j}),(p_j))BL((B_{H/V,j}),(p_j)) \tag{5}$$

where $B_{V,j} \colon V \to B_j V$ is the restriction of B_j to V and $B_{H/V,j} \colon H/V \to H_j/B_j V$ is the quotient map defined by $B_{H/V,j}(x + V) = B_j x + B_j V$.

We wish to apply the induction hypothesis to determine $BL((B_{V,j}),(p_j))$ and $BL((B_{H/V,j}),(p_j))$. Note that both V and H/V are of dimension less that $\dim H$. For each j we can decompose B_j with respect to the decompositions $H = V \oplus V^\perp$ and $H_j = B_jV \oplus (B_jV)^\perp$ as

$$B_j = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \tag{6}$$

where $A = B_{V,j}$ is surjective. Then

$$\mathrm{Id}_{H_j} = B_j B_j^* = \begin{pmatrix} AA^* + BB^* & BC^* \\ CB^* & CC^* \end{pmatrix}$$

so $AA^* + BB^* = \mathrm{Id}_{B_jV}$ and $CC^* = \mathrm{Id}_{(B_jV)^\perp}$. Since $V^\perp$ is isometric to H/V and $(B_jV)^\perp$ is isometric to $H_j/(B_jV)$ we see from this that $B_{H/V,j}B^*_{H/V,j} = \mathrm{Id}_{H_j/B_jV}$ and that $B_{V,j}B^*_{V,j} \le \mathrm{Id}_{B_jV}$ in the sense of positive definite operators.

Thus the induction hypothesis shows directly that $BL((B_{H/V,j}),(p_j)) \ge 1$. For $((B_{V,j}),(p_j))$ we need to normalise. Since $B_{V,j}$ is surjective we get that $B_{V,j}B^*_{V,j}$ is a positive definite operator. We can therefore define $\tilde{B}_{V,j} = (B_{V,j}B^*_{V,j})^{-1/2}B_{V,j}$ and note that $\tilde{B}_{V,j}\tilde{B}^*_{V,j} = \mathrm{Id}_{B_jV}$. The formulas for equivalence of Brascamp–Lieb constants, see Lemma 3.3 of [3] give that

$$BL((B_{V,j}),(p_j)) = BL((\tilde{B}_{V,j}),(p_j)) \prod_{j=1}^m \det(B_{V,j}B^*_{V,j})^{-p_j/2}$$

so by the induction hypothesis and the fact that $\det(B_{V,j}B^*_{V,j}) \le 1$ we see that

$$BL((B_{V,j}),(p_j)) \ge 1.$$

Collecting this we see that if $((B_j),(p_j))$ has a critical subspace then

$$BL((B_j),(p_j)) \ge 1.$$

For the direction of Theorem 5 stating that equality implies geometricity we need to study the cases of equality in the preceeding inequality. Equation (5) shows that for equality to hold, both $((B_{V,j}),(p_j))$ and $((B_{H/V,j}),(p_j))$ must be geometric by the induction hypothesis. Furthermore we must have $\det(B_{V,j}B^*_{V,j}) = 1 = \det(\mathrm{Id}_{B_jV})$ for all j which in turn forces $B = 0$ where B is the block from equation (6) for B_j. This means that each B_j splits into an operator $V \to B_jV$ and an operator $V^\perp \to (B_jV)^\perp$ so that in order to show the geometricity of $((B_j),(p_j))$ it is enough to verify the geometricity condition on V and $V^\perp$ separately and there it follows from the geometricity of $((B_{V,j}),(p_j))$ and $((B_{H/V,j}),(p_j))$ respectively.

In E_3 which is the remaining portion of E the Brascamp–Lieb inequality holds and $((B_j),(p_j))$ has no critical subspaces. Note that E_3 is not closed but any convergent sequence from E_3 will have a limit in E. So consider a convergent sequence $(B_{\nu,j})$ so that $(B_{0,j}) = \lim_{\nu\to\infty}(B_{\nu,j})$ exists. If the Brascamp–Lieb inequality does not hold for $((B_{0,j}),(p_j))$, i.e. $(B_{0,j}) \in E_1$, or the datum $((B_{0,j}),(p_j))$ has a critical subspace, i.e. $(B_{0,j}) \in E_2$, then we note that there exists a gaussian input (A_j) such that $BL((B_{0,j}),(p_j),(A_j)) > 1-\epsilon$ for any $\epsilon > 0$. Since $BL((B_j),(p_j),(A_j))$ is a continuous function of (B_j) we see that we can find an $N > 0$ such that if $\nu > N$ then $BL((B_{\nu,j}),(p_j),(A_j)) > 1-\epsilon$.

From this we see that for (B_j) close to the boundary of E_3 relative to E we have that $BL((B_j),(p_j)) > 1-\epsilon$. Thus if there exists a (B_j) such that $BL((B_j),(p_j)) < 1-\epsilon$ it would have to be in the interior of E_3.

We will now show that $BL((B_j),(p_j))$ is a differentiable function of (B_j) in all of E_3, so in particular E_3 is open relative to E. Furthermore, we will find that the critical points of this function occur only when $((B_j),(p_j))$ is geometric and then it is known that $BL((B_j),(p_j)) = 1$. Together with the remarks of the last paragraph, this proves the theorem.

The rest of the proof consists of two lemmas verifying the claims of the previous paragraph.

Lemma 7. *The only critical points of the function*

$$K((B_j),(A_j)) = \log\left(\frac{\prod_{j=1}^m (\det A_j)^{p_j}}{\det(\sum_{j=1}^m p_j B_j^* A_j B_j)}\right)^{\frac{1}{2}} \tag{7}$$

$$= \frac{1}{2}\sum_{j=1}^m p_j \log\det A_j - \frac{1}{2}\log\det(\sum_{j=1}^m p_j B_j^* A_j B_j) \tag{8}$$

in the interior of E_3 *occur when* $((B_j),(p_j))$ *is a geometric datum.*

Proof. Let us consider the perturbations $A_j+\epsilon Q_j$ and $B_j+\epsilon T_j$ where $Q_j : H_j \to H_j$ is symmetric and $T_j : H \to H_j$ is such that $B_j T_j^* = 0$. The last condition ensures that up to order $O(\epsilon^2)$ we have that $B_j + \epsilon T_j$ satisfies the normalising condition $(B_j + \epsilon T_j)(B_j + \epsilon T_j)^* = \mathrm{Id}_{H_j}$.

Now, $((B_j),(A_j))$ is a critical point for the function K if and only if the expansion of $K((B_j + \epsilon T_j),(A_j + \epsilon Q_j))$ in powers of ϵ contains no ϵ^1 terms.

It is established in [3] that the ϵ terms in the perturbation of $(A_j + \epsilon Q_j)$ cancel out provided that

$$A_j^{-1} = B_j M^{-1} B_j^* \tag{9}$$

where

$$M = \sum_{j=1}^m p_j B_j^* A_j B_j. \tag{10}$$

Let us therefore consider

$$\begin{aligned}
&\frac{d}{d\epsilon}\left(\frac{1}{2}\sum_{j=1}^m p_j \log\det A_j - \frac{1}{2}\log\det(\sum_{j=1}^m p_j (B_j+\epsilon T_j)^* A_j (B_j+\epsilon T_j))\right)\\
&= -\frac{1}{2}p_j \operatorname{tr}(M^{-1}(B_j^* A_j T_j + T_j^* A_j B_j))\\
&= -\frac{1}{2}p_j (\operatorname{tr}(M^{-1}B_j^* A_j T_j) + \operatorname{tr}(M^{-1}T_j^* A_j B_j)) = -p_j \operatorname{tr}(A_j T_j M^{-1} B_j^*).
\end{aligned}$$

This derivative is 0 provided that $\operatorname{tr}(A_j T_j M^{-1} B_j^*) = 0$.

Fix j and consider the decomposition $H = B_j^* H_j \oplus (B_j^* H_j)^\perp$. Take e_j in H_j and define $u_j = B_j^* e_j \in B_j^* H_j$ and furthermore take $v_j \in (B_j^* H_j)^\perp$. Consider $T_j = A_j^{-1} e_j v_j^*$. Then $B_j T_j^* = B_j v_j e_j^* A_j^{-1} = 0$ since $B_j v_j = 0$ by the definition of v_j. Then

$$\operatorname{tr}(A_j T_j M^{-1} B_j^*) = \operatorname{tr}(v_j^* M^{-1} B_j^* e_j) = \langle M^{-1} u_j, v_j\rangle.$$

Thus we see that a necessary condition for a stationary point of K is that for every j we have that for every $u_j \in B_j^* H_j$ and $v_j \in (B_j^* H_j)^\perp$ that $\langle M^{-1}u_j, v_j\rangle = 0$. From this it is clear that we can decompose M^{-1} as the tensor product of an operator acting on $B_j^* H_j$ and an operator acting on $(B_j^* H_j)^\perp$. The same will be true for M, that is $M = M_{j0} \otimes M_{j1}$ where M_{j0} is an operator on $B_j^* H_j$ and M_{j1} an operator on $(B_j^* H_j)^\perp$.

Consider an eigenvalue λ of M and the corresponding eigenspace E_λ. We see that $E_\lambda = E_{\lambda j0} \oplus E_{\lambda j1}$ where $E_{\lambda j0}$ is the eigenspace of λ for M_{j0} and and $E_{\lambda j1}$ is the eigenspace of λ for M_{j1}. Furthermore, $E_{\hat\lambda} = E_{\hat\lambda j0} \oplus E_{\hat\lambda j1}$ where the hat denotes the eigenspace of all other eigenvalues. Now note that $H = E_\lambda \oplus E_{\hat\lambda}$ and since B_j acts bijectively from $B_j^* H_j$ to H_j we get that $B_j E_\lambda = B_j E_{\lambda j0}$ and $B_j E_{\hat\lambda} = B_j E_{\hat\lambda j0}$ and $B_j E_\lambda \cap B_j E_{\hat\lambda} = \{0\}$ and $B_j E_\lambda + B_j E_{\hat\lambda} = H_j$. This shows that E_λ and $E_{\hat\lambda}$ form a critical pair in the sense of Definition 7.3 of [3] and thus that

E_λ is a critical subspace. Since we are assuming that H has no critical subspace we get that M has only a single eigenvalue and is thus a multiple of the identity operator, $M = \lambda \operatorname{Id}_H$. Then $A_j = \lambda \operatorname{Id}_{H_j}$ follows from (9) and then $\operatorname{Id}_H = \sum_{j=1}^m p_j B_j^* B_j$ follows from (10). Thus $((B_j),(p_j))$ is a geometric datum.

Also

$$K((B_j),(A_j)) = \log\left(\frac{\lambda^{\sum_{j\in J} p_j}}{\det_{E_\lambda} M}\right) = \log\left(\frac{\lambda^{\sum_{j\in J} p_j}}{\lambda^{\dim E_\lambda}}\right) = 0$$

where the last equality follows from the relationship $\sum_{j\in J} p_j = \dim E_\lambda$ which in turn follows from the criticality of E_λ. □

The final step in the proof of the theorem is the following differentiability lemma.

Lemma 8. *If the datum $((B_j),(p_j))$ is extremisable and has no critical subspaces then in a neighbourhood of this datum, the solution (A_j) to the optimisation problem* (3) *is a differentiable function of (B_j).*

Remark 9. Since $K((B_j),(A_j))$ is invariant under the scaling $(A_j) \mapsto (\lambda A_j)$, we should properly think of the solution to the optimisation problem as an element of the quotient space arrived at after dividing out this invariance.

Remark 10. This lemma implies the openness of E_3 relative to E since we know that if the optimisation problem (3) has a solution for $((B_j),(p_j))$ then this is an extremisable datum. Furthermore, if an extremisable datum $((B_j),(p_j))$ has a critical subspace V then V will be part of a critical pair (V,W) and the solution (A_j) to the optimisation problem cannot be a differentiable function of (B_j) in a neighbourhood of the datum since (A_j) will split into an operator acting on V and another one acting on W. We can apply different scalings to each of the suboperators and find a new solution $(\tilde{A}_j)$ to the optimisation problem which is arbitrarily close to (A_j) but is not related to (A_j) by a simple scaling $\tilde{A}_j = \lambda A_j$.

Proof. Introduce the functions $S_{j'}((B_j),(A_j)) = p_{j'} A_{j'}^{-1} - p_{j'} B_{j'} M^{-1} B_{j'}^*$. Then (A_j) is a solution to the optimisation problem if and only if

$$S_{j'}((B_j),(A_j)) = 0 \tag{11}$$

for every j'. We wish to use the implicit function theorem to show that the solution of this system of equation is a differentiable function of (B_j).

Our aim will be to show that the first derivative of $S = (S_j)$ with respect to (A_j) at a solution to (11) is not zero except in the direction of (A_j).

The derivative of (S_j) at (A_j) in the direction of $(Q_j) \in \oplus_{j=1}^m \operatorname{Symm}(H_j)$ is given by

$$D(Q_j) = D_{(A_j)}(S_j)(Q_j) = \left(-p_j A_j^{-1} Q_j A_j^{-1} + \sum_{j'=1}^m p_j p_{j'} B_j M^{-1} B_{j'}^* Q_{j'} B_{j'} M^{-1} B_j^*\right).$$

Note that $D_{(A_j)}(S_j)(Q_j) \in \oplus_{j=1}^m \operatorname{Symm}(H_j) \subset \operatorname{Symm}(\oplus_{j=1}^m H_j)$. Let us take $R = (R_j) \in \oplus_{j=1}^m \operatorname{Symm}(H_j)$ and calculate $\langle D(A_j^{\frac12} R_j A_j^{\frac12}), A_j^{\frac12} R_j A_j^{\frac12})\rangle$ where $\langle\cdot,\cdot\rangle$ is the Frobenius inner product on $\operatorname{Symm}(\oplus_j H_j)$. This equals

$$-\sum_j \operatorname{tr}(R_j R_j) + \sum_j \operatorname{tr}\left(\sum_{j'} p_j p_{j'} P_{jj'} R_{j'} P_{j'j} R_j\right)$$

where $P_{j'j} = p_j^{\frac12} p_{j'}^{\frac12} A_j^{\frac12} B_j M^{-\frac12} M^{-\frac12} B_{j'}^* A_{j'}^{\frac12}$

Next, introduce the transformation

$$T = \begin{pmatrix} --- & p_1^{\frac{1}{2}} A_1^{\frac{1}{2}} B_1 M^{-\frac{1}{2}} & --- \\ & \vdots & \\ --- & p_m^{\frac{1}{2}} A_m^{\frac{1}{2}} B_m M^{-\frac{1}{2}} & --- \end{pmatrix} \in L(H, \oplus_j H_j)$$

and note that $T^*T = M^{-\frac{1}{2}} \left(\sum_j p_j B_j^* A_j B_j \right) M^{-\frac{1}{2}} = \mathrm{Id}_H$ by the definition of M. Thus $P = TT^*$ is a projection transformation and we see that P is composed of the blocks $P_{jj'}$ as defined above, $P = (P_{jj'})$. Then

$$\langle D(A_j^{\frac{1}{2}} R_j A_j^{\frac{1}{2}}), A_j^{\frac{1}{2}} R_j A_j^{\frac{1}{2}}) \rangle = -\operatorname{tr}(RPR) + \operatorname{tr}(PRPR) = \operatorname{tr}((P-I)RPR)$$

where we have reinterpreted R as a block diagonal element of $\mathrm{Symm}(H_j)$, written $I = \mathrm{Id}_{\oplus_j H_j}$ and noted that the diagonal blocks P_{jj} of P equal $p_j \,\mathrm{Id}_{H_j}$ since $B_j M^{-1} B_j^* = A_j^{-1}$.

Since P and $I - P$ are projection transformations we can calculate further

$$-\operatorname{tr}((I-P)RPR) = -\operatorname{tr}((I-P)RPPR(I-P)) = -\|(I-P)RP\|$$

where $\|\cdot\|$ is the norm associated to the inner product discussed above. We are interested in the condition

$$0 = \langle D(A_j^{\frac{1}{2}} R_j A_j^{\frac{1}{2}}), A_j^{\frac{1}{2}} R_j A_j^{\frac{1}{2}}) \rangle = \|(I-P)RP\|.$$

For this to hold, R must map the image of T to the image of T, in other words, for each $x \in H$ there is a $y \in H$ such that $RTx = Ty$. Since $T^*T = \mathrm{Id}_H$ we see that $y = T^*RTx$ and by considering each block in the matrix equation we see that $\tilde{B}_j y = R_j \tilde{B}_j x$ where $\tilde{B}_j = A_j^{\frac{1}{2}} B_j M^{-\frac{1}{2}}$. Note that $\tilde{B}_j \tilde{B}_j^* = \mathrm{Id}_{H_j}$ so that $\tilde{B}_j$ acts as a projection operator. We may thus assume that H_j is a subspace of H and $\tilde{B}_j$ is the orthogonal projection from H to H_j.

Now, note that T^*RT is a symmetric operator so its eigenspaces form an orthogonal decomposition of H. Let λ be an eigenvalue and consider an element x from the eigenspace E_λ. Then $y = T^*RTx = \lambda x$ and $\tilde{B}_j x$ is an eigenvector of R_j with eigenvalue λ for all j. If λ_1 and λ_2 are distinct eigenvalues of T^*RT with eigenspaces E_{λ_1} and E_{λ_2} then $\tilde{B}_j E_{\lambda_1}$ and $\tilde{B}_j E_{\lambda_2}$ are subspaces of the eigenspaces of R_j (in H_j) with the corresponding eigenvalues. Since R_j is symmetric, its eigenspaces are orthogonal to each other and we conclude that each eigenspace of T^*RT is a critical space for H. Since we are assuming that H has no critical subspaces we thus get that $T^*RT = \lambda \,\mathrm{Id}_H$. This implies that $R = \lambda \,\mathrm{Id}_{H_j}$ so in fact $D(Q_j)$ is only zero when Q_j is a multiple of A_j.

This completes the proof of the lemma. □

References

[1] Keith Ball, *Volumes of sections of cubes and related problems*, Geometric aspects of functional analysis (1987–88), Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, pp. 251–260. MR MR1008726 (90i:52019)

[2] Franck Barthe, *On a reverse form of the Brascamp–Lieb inequality*, Invent. Math. **134** (1998), no. 2, 335–361. MR MR1650312 (99i:26021)

[3] Jonathan Bennett, Anthony Carbery, Michael Christ, and Terence Tao, *The Brascamp–Lieb inequalities: finiteness, structure, and extremals*, Geom. Funct. Anal., to appear.

[4] Herm J. Brascamp and Elliott H. Lieb, *Best constants in Young's inequality, its converse, and its generalization to more than three functions*, Advances in Math. **20** (1976), no. 2, 151–173. MR MR0412366 (54 #492)

[5] Eric A. Carlen and Dario Cordero-Erausquin, *Subadditivity of the entropy and its relation to brascamp-lieb type inequalities*, 2007.

[6] Eric A. Carlen, Elliott H. Lieb, and Michael Loss, *A sharp analog of Young's inequality on S^N and related entropy inequalities*, J. Geom. Anal. **14** (2004), no. 3, 487–520. MR MR2077162 (2005k:82046)

[7] Elliott H. Lieb, *Gaussian kernels have only Gaussian maximizers*, Invent. Math. **102** (1990), no. 1, 179–208. MR MR1069246 (91i:42014)

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