OPTIMISERS FOR THE BRASCAMP–LIEB INEQUALITY

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Abstract. We find all optimisers for the Brascamp–Lieb inequality, thus completing the problem which was settled in special cases by Barthe; Carlen, Lieb and Loss; and Bennett, Carbery, Christ and Tao. Our approach to the solution is based on the heat flow methods introduced by the second and third sets of authors above. We present the heat flow method in the form which is most appropriate for our study and also expand the structural theory for the Brascamp–Lieb inequality as necessary for the description of the optimisers.

1. Introduction

The Brascamp–Lieb inequality unifies and generalises several of the most central inequalities in analysis, among others the inequalities of Hölder, Young and Loomis–Whitney. It has the form

$$\int_H \prod_{j=1}^m f_j^{p_j}(B_j x)\,\mathrm{d}x \leq C \prod_{j=1}^m \left(\int_{H_j} f_j \right)^{p_j} \tag{1}$$

where H and H_j are finite dimensional Hilbert spaces of dimensions n and n_j respectively, $B_j : H \to H_j$ are linear maps, p_j are non-negative numbers, C is a finite constant and f_j are non-negative functions. We shall refer to $((B_j),(p_j))$ as the Brascamp–Lieb datum for this inequality.

The inequality was first written down by Brascamp and Lieb in [6] where they pose two questions. The first one is to find the necessary and sufficient conditions on the datum $((B_j),(p_j))$ for (1) to hold and the second one is to determine when the best constant for (1) is attained by a tuple of centred gaussian functions, $f_j(x) = e^{-\langle x, A_j x\rangle}$ with each A_j a symmetric and positive semi-definite linear transformation.

In [8] Lieb showed that gaussians exhaust the inequality in the following sense.

Theorem 1 (Lieb's Theorem)**.** *Let $C((B_j),(p_j))$ be the smallest constant we can take in* (1) *so that it holds for all tuples (f_j) of integrable functions and let $C_g((B_j),(p_j))$ be the smallest constant we can take so that it holds for tuples of centred gaussians. Then*

$$C((B_j),(p_j)) = C_g((B_j),(p_j)). \tag{2}$$

Brascamp and Lieb proved this theorem in the case when each B_j has rank one already in [6].

Even with Lieb's Theorem, several questions remained open: when is the constant $C((B_j),(p_j))$ finite, what is its value, when is this value attained and what are all the functions that attain it? A practical case which was first studied by Ball and later generalised by Barthe is that of a geometric datum.

Definition 2. We say that a Brascamp–Lieb datum is *geometric* if $B_j B_j^* = \mathrm{Id}_{H_j}$ for each j and

$$\sum_{j=1}^{m} p_j B_j^* B_j = \mathrm{Id}_H \tag{3}$$

In this case it was shown by Ball [1] and Barthe [2] that (1) holds with $C = 1$ and that the tuple $(f_j) = (e^{-\langle\cdot,\cdot\rangle})$ is an extremiser which does attain that constant.

Barthe [2] also gave necessary and sufficient conditions for (1) to hold in the case when each B_j has rank one and in the case when a gaussian extremiser exists he determined all the functions such that (1) holds with equality. This was later re-examined by Carlen, Lieb and Loss [7] who introduced heat flow arguments to the theory of Brascamp–Lieb inequalities. They also showed that the existence of extremisers implies the existence of gaussian extremisers.

The general case was not settled until the two papers of Bennett, Carbery, Christ and Tao, [5] and [4]. They give necessary and sufficient conditions for (1) to hold. These are the following:

$$p_j \geq 0 \tag{4}$$

for all j,

$$\dim H = \sum_j p_j \dim H_j, \tag{5}$$

and

$$\dim V \leq \sum_j p_j \dim B_j V \tag{6}$$

for all subspaces V of H.

Let us now return to the question of when (1) is extremisable, that is, for which Brascamp–Lieb data $((B_j), (p_j))$ does there exist a tuple (f_j) of functions such that (1) holds with equality with the constant $C((B_j), (p_j))$. Again, the general case is settled in [5]; in the rank one case this goes back to [7]. To state the result we introduce the concept of equivalence.

Definition 3. Let $B_j : H \to H_j$ and $B_j' : H' \to H_j'$ be linear transformations. We say that (B_j) and (B_j') are *equivalent* if there exist invertible linear transformations $C : H' \to H$ and $C_j : H_j' \to H_j$ such that $B_j' = C_j^{-1} B_j C$.

By a simple change of variables we can see that if (f_j) is an extremiser for $((B_j), (p_j))$ then $(f_j') = (f_j \circ C_j)$ is an extremiser for $((B_j'), (p_j))$ and it is clear that if each function in the tuple (f_j) is a gaussian then the same holds for each function in the tuple (f_j').

In [5] the following is proved.

Theorem 4. *If $((B_j), (p_j))$ is extremisable then it is gaussian extremisable. Furthermore, any extremisable datum is equivalent to a geometric datum.*

In this paper we address the question of finding all extremisers for the Brascamp–Lieb inequality. This is a problem that has been settled in the rank one case in [2] and [7]. In [5] an analysis of the general case is started but it only addresses the rather specific situation when the spaces $B_j^* H_j$ are pairwise disjoint, except at the origin.

Of course, several basic cases have been known for a long time. One instance of the Brascamp–Lieb inequality is Hölder's inequality

$$\int f^p(x) g^q(x)\,\mathrm{d}x \le \left(\int f(x)\,\mathrm{d}x\right)^p \left(\int g(x)\,\mathrm{d}x\right)^q, \tag{7}$$

where $p+q=1$. For $p,q>0$, there is equality here if and only if there exists an integrable function u such that $f=c_1 u$ and $g=c_2 u$ where c_1 and c_2 are constants.

Another example of a Brascamp–Lieb inequality for which the extremisers are known is the Loomis–Whitney inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j^{\frac{1}{n-1}}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)\,\mathrm{d}x \le \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} f_j\right)^{\frac{1}{n-1}}. \tag{8}$$

Here it is known that the tuple (f_j) is an extremiser if and only if there exist integrable functions $u_1,\dots,u_n$ of one variable such that

$$f_j(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n) = c_j u_1(x_1)\cdots u_{j-1}(x_{j-1}) u_{j+1}(x_{j+1})\cdots u_n(x_n).$$

where c_j are constants.

In other cases the shape of the the extremisers can be much more restricted. An example is Young's inequality

$$\int f^p(x) g^q(y) h^r(x-y)\,\mathrm{d}x\,\mathrm{d}y \le c_{p,q,r}\left(\int f\right)^p \left(\int g\right)^q \left(\int h\right)^r \tag{9}$$

for $p+q+r=2$ and $0\le p,q,r\le 1$ where $c_{p,q,r}$ is the sharp constant found by Beckner in [3]. He also showed that for $p,q,r<1$ this inequality has gaussian extremisers. Brascamp and Lieb [6] reproved this in a different way and showed that the only extremisers are gaussians of a specific form.

In this paper, we find the form of all optimisers for the Brascamp–Lieb inequality in all cases. Although the theorem is slightly technical to state, see Theorems 12 and 15 below, the content can be summarised by saying that there is nothing weird going on which does appear in the above three cases.

More explicitly, for any optimiser, there will exist a decomposition of H as a direct sum of subspaces. On each of these subspaces there is an underlying function and each f_j in the optimiser tuple is the product of the relevant underlying functions and some arbitrary constant. Furthermore, each of the underlying functions can either be any integrable function or must be a gaussian of a certain shape.

The structure of this paper is as follows. In Section 2 we will set up the heat flow and derive the monotonicity formula which proves the Brascamp–Lieb inequality. In Section 3 we study the structural theory of Brascamp–Lieb data. Understanding of this structure is necessary to give a precise description of the optimisers. Finally, in Section 4 we study the monotonicity formula carefully and determine all possible forms of the optimisers.

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2. Monotonicity properties

In this section, we will use a heat flow argument along the lines set forth in [7] and [5] to derive a monotonicity formula which will be the basis of our study of the optimisers. This argument can be carried out for general extremisable datum, but since the argument in the following sections will mostly focus on the case of geometric datum we will concentrate on that case here.

So, let (B_j, p_j) be a geometric Brascamp–Lieb datum and further assume that H_j is a subspace of H and that $B_j : H \to H_j$ is the orthogonal projection. (We note that this condition on B_j does not restrict the applicability of the results as any surjective linear map $H \to H_j$ can be written as a composition of such a B_j and an invertible linear map from H_j to itself.)

Let f_j be non-negative Schwartz functions on H_j for $j = 1, \ldots, m$. Let $\tilde{f}_j(x,t)$ for $t \geq 0$ be the solution to the initial value problem

$$\begin{aligned} \tilde{f}_j(x,0) &= f_j(B_j x) \\ \frac{\partial}{\partial t}\tilde{f}_j(x,t) &= \Delta \tilde{f}_j(x,t) \qquad t > 0 \end{aligned} \tag{10}$$

where $\Delta \tilde{f}_j = \operatorname{div}(\nabla \tilde{f}_j)$.

Let us define the product

$$F(x,t) = \prod_{j=1}^{m} \tilde{f}_j^{\,p_j}(x,t).$$

Our aim is to discuss monotonicity properties of $\int F$, that is, under what circumstances we can say that

$$\frac{\partial}{\partial t} \int F(x,t)\,\mathrm{d}x \geq 0.$$

From the definitions we see that

$$\frac{\partial}{\partial t} F = F \sum_{j=1}^{m} p_j \frac{\Delta \tilde{f}_j}{\tilde{f}_j} = F \sum_{j=1}^{m} p_j \frac{\operatorname{div}(v_j \tilde{f}_j)}{\tilde{f}_j}$$

where $v_j = \frac{\nabla \tilde{f}_j}{\tilde{f}_j}$. We calculate further

$$\frac{\partial}{\partial t} F = F \sum_{j=1}^{m} p_j \left(\langle \frac{\nabla \tilde{f}_j}{\tilde{f}_j}, v_j \rangle + \operatorname{div}(v_j) \right) = I + II.$$

Note that

$$\int I = \int F \sum_{j=1}^{m} p_j \langle v_j, v_j \rangle.$$

In the literature concerning this problem on $\mathbb{R}^n$ and S^n there have been several ideas put forward on how to proceed from this point. In [5] the authors proceed on $\mathbb{R}^n$ by using the log-concavity of the heat kernel. Another approach which appears in [7] is to use integration by parts for I and write

$$\begin{aligned}\int II &= \int F \sum_{j=1}^m p_j \operatorname{div}(v_j) = \sum_{j=1}^m p_j \int F \operatorname{div}(v_j)\\ &= \sum_{j=1}^m p_j \left(\int \operatorname{div}(F v_j) - \int \langle \nabla F, v_j \rangle \right) = -\int F \langle \sum_{j'=1}^m p_{j'} v_{j'}, \sum_{j=1}^m p_j v_j \rangle\end{aligned}$$

where we have used the divergence theorem to eliminate the first term in the next to last expression. This is justified for any $t > 0$ since H is boundaryless, the integrand has enough smoothness and decays rapidly at infinity. The proof of this last statement is in Lemma 5 at the end of this section.

Let us show that

$$\tilde{f}_j(x,t) = \int_H \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\|x-y\|^2}{4t}} f_j(B_j y)\,\mathrm{d}y.$$

is a solution to (10). It is a straightforward calculation which shows that the relationship between the derivatives holds so the only thing we need to check is that $\tilde{f}_j(x,0) = f_j(B_j x)$. With the decomposition $H = H_j \oplus H_j^\perp$ we can write $\tilde{f}_j(x,t)$ as

$$\int_{H_j} \int_{H_j^\perp} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\|B_j x - y_1\|^2 + \|B_j^\perp x - y_2\|^2}{4t}} f_j(y_1)\,\mathrm{d}y_2\,\mathrm{d}y_1$$

where $B_j^\perp$ denotes the projection onto the orthogonal complement of H_j. In the inner integral we make the change of variables $y_2 \mapsto y_2 - B_j^\perp x$ and get

$$\int_{H_j} \left(\int_{H_j^\perp} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\|y_2\|^2}{4t}}\,\mathrm{d}y_2 \right) e^{-\frac{\|B_j x - y_1\|^2}{4t}} f_j(y_1)\,\mathrm{d}y_1. \tag{11}$$

By carrying out the y_2 integration we get

$$\int_{H_j} \frac{1}{(4\pi t)^{\frac{n_j}{2}}} e^{-\frac{\|B_j x - y_1\|^2}{4t}} f_j(y_1)\,\mathrm{d}y_1. \tag{12}$$

Let us call the kernel in this expression $K_{j,t}(B_j x - y_1)$ so that the whole integral is $(K_{j,t} * f_j)(B_j x)$. We note that $K_{j,t}$ is an approximation to the identity on H_j and f_j is integrable so we see that $\lim_{t\to 0}(K_{j,t} * f_j)(B_j x) = f_j(B_j x)$. This confirms that $\tilde{f}_j(x,t)$ is a solution to (10). Furthermore, we see that $\tilde{f}_j(x,t)$ depends only on $B_j x$ so we can write $\tilde{f}_j(x,t)$ as $f_j(B_j x, t)$.

We will now calculate the limit of $\int F(x,t)\,\mathrm{d}x$ as t tends to infinity. We have that

$$\begin{aligned}\int_H \prod_{j=1}^m \tilde{f}_j^{\,p_j}(x,t)\,\mathrm{d}x &= \int_H \prod_{j=1}^m \tilde{f}_j^{\,p_j}(t^{\frac{1}{2}}w,t)t^{\frac{n}{2}}\,\mathrm{d}w \\ &= \int_H \prod_{j=1}^m \left(t^{\frac{n_j}{2}}\tilde{f}_j(t^{\frac{1}{2}}w,t)\right)^{p_j}\,\mathrm{d}w\end{aligned}$$

where we have made the change of variables $x = t^{\frac{1}{2}}w$ and used the necessary condition (5). From (12) we see that we can write the quantity within the parentheses as

$$\int_{H_j} \frac{1}{(4\pi)^{\frac{n_j}{2}}} e^{-\frac{1}{4}\|B_j w - t^{-\frac{1}{2}}y_1\|^2} f_j(y_1)\,\mathrm{d}y_1.$$

Thus by dominated convergence which is applicable as $\cap_j \ker B_j = \{0\}$ since the datum is gaussian extremisable we get that

$$\lim_{t\to\infty}\int_H \prod_{j=1}^m \tilde{f}_j^{\,p_j}(x,t)\,\mathrm{d}x = \int_H \prod_{j=1}^m L_j^{p_j}(w)\,\mathrm{d}w \prod_{j=1}^m \left(\int_{H_j} f_j\right)^{p_j}$$

where

$$L_j(w) = \frac{1}{(4\pi)^{\frac{n_j}{2}}} e^{-\frac{1}{4}\|B_j w\|^2}.$$

We can evaluate

$$\int_H \prod_{j=1}^m L_j^{p_j}(w)\,\mathrm{d}w = \int_H \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{1}{4}\langle \sum_j p_j B_j^* B_j w, w\rangle}\,\mathrm{d}w = 1$$

since $n = \sum_j p_j n_j$ by the scaling condition (5) and $\sum_j p_j B_j^* B_j = \mathrm{Id}_H$ by the geometricity. We thus conclude that

$$\lim_{t\to\infty}\int_H \prod_{j=1}^m \tilde{f}_j^{\,p_j}(x,t)\,\mathrm{d}x = \prod_{j=1}^m \left(\int_{H_j} f_j\right)^{p_j}. \tag{13}$$

In what follows we shall fix a time $t > 0$ and mostly not write it explicitly in the equations. We get that

$$\frac{\nabla \tilde{f}_j(x)}{\tilde{f}_j(x)} = \frac{\nabla(f_j\circ B_j)(x)}{(f_j\circ B_j)(x)} = \frac{B_j^*\nabla f_j(B_j x)}{f_j(B_j x)} = B_j^*\nabla h_j(B_j x)$$

where $h_j = \log f_j$.

We have that

$$\frac{\partial}{\partial t}\int F(x,t)\,\mathrm{d}x = \int F(x,t)\Xi(x,t)\,\mathrm{d}x$$

where

$$\Xi(x) = \sum_{j=1}^{m} p_j \langle B_j^* \nabla h_j(B_j x), B_j^* \nabla h_j(B_j x)\rangle$$
$$- \left\langle \sum_{j'=1}^{m} p_{j'} B_{j'}^* \nabla h_{j'}(B_{j'} x), \sum_{j=1}^{m} p_j B_j^* \nabla h_j(B_j x) \right\rangle$$

and our aim is to show that this quantity is non-negative. That will make the integrand $F\Xi$ pointwise non-negative.

We can write this quantity in a more succinct form by introducing some notation.

Let $T : H \to \oplus_j H_j$[1] be the transformation

$$T = \begin{pmatrix} --- & p_1^{\frac{1}{2}} B_1 & --- \\ & \vdots & \\ --- & p_m^{\frac{1}{2}} B_m & --- \end{pmatrix}$$

and $A(x) \in \oplus_j H_j$ be given by

$$A(x) := \begin{pmatrix} p_1^{\frac{1}{2}} \nabla h_1(B_1 x) \\ \vdots \\ p_m^{\frac{1}{2}} \nabla h_m(B_m x) \end{pmatrix}.$$

Then we see that

$$\Xi(x) = \langle A(x), A(x)\rangle - \langle T^* A(x), T^* A(x)\rangle.$$

Note that $T^*T = \sum_j p_j B_j^* B_j = \mathrm{Id}_H$ by the geometricity assumption so $P = TT^* = T(T^*T)^{-1}T^*$ is a projection transformation.

We can thus write $\Xi(x)$ as

$$\Xi(x) = \langle A(x), A(x)\rangle - \langle TT^* A(x), A(x)\rangle = \langle (I - P)A(x), A(x)\rangle \tag{14}$$

where I is the relevant identity transformation. Since P is a projection transformation it is clear that $\Xi \geq 0$ and we have therefore shown that

$$\frac{\partial}{\partial t} \int F(x,t)\,\mathrm{d}x \geq 0 \tag{15}$$

for all $t > 0$.

We have thus established that $\int F(x, t_1) \leq \int F(x, t_2)$ for any $0 < t_1 \leq t_2$. We know from the discussion above that

$$\lim_{t\to 0} F(x,t) = \prod_{j=1}^{m} f_j^{p_j}(B_j x)$$

so by Fatou's lemma we get that

$$\int_H \prod_{j=1}^{m} f_j^{p_j}(B_j x)\,\mathrm{d}x = \int_H F(x,0)\,\mathrm{d}x \leq \int_H F(x,t)\,\mathrm{d}x$$

[1] This direct sum is over independent copies of H_j.

for all $t > 0$.

By comparing the limits as t tends to zero and infinity we thus arrive at the inequality

$$\int_H \prod_{j=1}^{m} f_j^{p_j}(B_j x)\,\mathrm{d}x \le \prod_{j=1}^{m}\left(\int_{H_j} f_j\right)^{p_j}. \tag{16}$$

2.1. **Size conditions.** Let us prove the technical lemma left behind.

Lemma 5. $\frac{\nabla f_j(x_j)}{f_j(x_j)}$ *has linear growth in* $x_j = B_j x$.

Proof. From (12) we see that

$$\frac{\nabla f_j(x_j)}{f_j(x_j)} = -\frac{1}{2t}\frac{\int_{H_j}(x_j - y)e^{-\frac{1}{4t}\|x_j-y\|^2} f_j(y)\,\mathrm{d}y}{\int_{H_j} e^{-\frac{1}{4t}\|x_j-y\|^2} f_j(y)\,\mathrm{d}y.} \tag{17}$$

Since f_j is a positive Schwartz function we can find a C such that

$$\int_{\|y\|>C} \|y\| f_j(y)\,\mathrm{d}y \le \int_{\|y\|<C} f_j(y)\,\mathrm{d}y. \tag{18}$$

We may assume that $\|x_j\| > C$. We split the integral in the numerator of (17) in two parts according to whether $\|x_j - y\| \le 2\|x_j\|$ or $\|x_j - y\| > 2\|x_j\|$. The first integral we can estimate by

$$2\|x_j\| \int_{H_j} e^{-\frac{1}{4t}\|x_j-y\|^2} f_j(y)\,\mathrm{d}y$$

and the contribution to the fraction from this term is therefore linear in $\|x_j\|$.

For the second part we note that if $\|x_j - y\| > 2\|x_j\|$ then $\|x_j - y\| < 2\|y\|$ and since $\|x_j\| > C$ we also get that $\|y\| > C$ and that $\|x_j - y\| > \|x_j - a\|$ for any point a such that $\|a\| < C$. We now see that

$$\begin{aligned}
&\left\|\int_{\|x_j-y\|>2\|x_j\|}(x_j - y)e^{-\frac{1}{4t}\|x_j-y\|^2} f_j(y)\,\mathrm{d}y\right\| \\
&\le 2\int_{\|x_j-y\|>2\|x_j\|} \|y\| e^{-\frac{1}{4t}\|x_j-y\|^2} f_j(y)\,\mathrm{d}y
\end{aligned}$$

and since the factor $\|x_j - y\|^2$ in the exponent here is larger for any point in the set $\{y : \|x_j - y\| > 2\|x_j\|\}$ than for any point in the set $\{y : \|y\| \le C\}$ and the set $\{y : \|x_j - y\| > 2\|x_j\|\}$ is contained in the set $\{y : \|y\| > C\}$ we get by (18) that we can estimate the second integral by

$$\int_{\|y\|\le C} e^{-\frac{1}{4t}\|x_j-y\|^2} f_j(y)\,\mathrm{d}y.$$

We have thus shown that $v_j(x) = \frac{B_j^* \nabla f_j(x_j)}{f_j(x_j)}$ has at most linear growth in x. □

Since $F(x) = \prod_j f_j(B_j x)$ and each f_j is a Schwartz function and $\cap_j \ker B_j = \{0\}$ we see that F is a Schwartz function and thus $F v_j$ is rapidly decreasing. This establishes that the use of the divergence theorem above was justified.

3. Structural theory

To be able to state our results on the form of the optimisers we need some structural theory for the Brascamp–Lieb inequality. In the discussion that follows we shall assume that $p_j > 0$ for each j. This is a harmless assumption in the sense that any factor with power zero in (1) can be omitted from the inequality without affecting the value of the expressions on either side.

In [5] two notions of criticality are defined.

Definition 6.

(1) A subspace V of H is said to be *critical* if V is neither $\{0\}$ nor H and the inequality (6) holds with equality for V, so

$$\dim V = \sum_j p_j \dim B_j V. \tag{19}$$

(2) A pair of subspaces (V, W) of H is said to be a *critical pair* if V and W are complementary in H, so $V + W = H$ and $V \cap W = \{0\}$, and $B_j V$ and $B_j W$ are complementary in H_j for each j.

We see that if V and W are complementary in H then H decomposes as the direct sum $H = V \oplus W$. Among other things, the following lemma is proved concerning these notions of criticality in [5].

Lemma 7. *Let $((B_j), (p_j))$ be a Brascamp–Lieb datum such that* (5) *holds and* (6) *holds for any subspace V of H. Then we have the following:*

(1) *Each component of a critical pair is a critical subspace.*
(2) *If the datum is extremisable then for any critical subspace V of H there exists a complimentary subspace W of H such that (V, W) is a critical pair.*
(3) *If the datum is geometric then it is extremisable and we may take W to be the orthogonal complement of V in H.*

We will now extend the structural theory in directions relevant to the description of the optimisers. We will assume that we are working with a geometric datum as that will simplify the discussion considerably. A description of the optimisers in the general case can then easily be given in terms of the optimisers with equivalence and Theorem 4 in mind.

By the condition imposed on B_j we may assume that H_j, the image of B_j, is a subspace of H and that B_j is the orthogonal projection from H onto H_j. Also, we will take $B_j^\perp$ to be the orthogonal projection from H onto $H_j^\perp$ the orthogonal complement of H_j.

If V is a critical subspace then V is part of a critical pair (V, W) and H decomposes as $H = V \oplus W$. Furthermore, $((B_j|_V), (p_j))$ is a Brascamp–Lieb datum such that (5) holds (for V) and (6) holds for any subspace U of V. The same result holds for W.

We can thus repeat the splitting of H into critical subspaces until we arrive at a *maximal critical decomposition* where we write H as a sum of pairwise orthogonal spaces, each of which is critical and has no critical subspace.

We now make the following definition:

Definition 8. A subspace K of H will be said to be *independent* with respect to the geometric datum $((B_j),(p_j))$ if it is not $\{0\}$ and has the form

$$K = \bigcap_{j=1}^{m} H_j^a$$

where for each j, H_j^a is either H_j or $H_j^\perp$.

Clearly there are at most 2^m independent subspaces for any datum and any two distinct independent subspaces are orthogonal to one another. The following is then a sensible definition.

Definition 9. The *independent decomposition* of H is the decomposition

$$H = K_{\mathrm{ind}} \bigoplus K_{\mathrm{dep}} = \left(\bigoplus_{k=1}^{k_0} K_k \right) \bigoplus K_{\mathrm{dep}}$$

where $\{K_k | k = 1, \ldots, k_0\}$ is an enumeration of the independent subspaces of H and K_{dep} is the orthogonal complement of K_{ind}.

The following lemma establishes the relationship between the concepts of criticality and independence.

Lemma 10. *Let K be an independent subspace of H and V be a critical one. Then*

1. *K is also critical and*
2. *V can be decomposed as the direct sum $V = (V \cap K) \oplus (V \cap K^\perp)$ and these two spaces are critical if they are not $\{0\}$.*

Proof. We prove the first part by showing that $(K, K^\perp)$ is a critical pair. For any j there are two possibilities, either $K \subset H_j$ or $K \subset H_j^\perp$. If $K \subset H_j$ then we can write $K^\perp$ as the orthogonal sum of $\tilde{K}$ and $H_j^\perp$ where $\tilde{K}$ is the orthogonal complement of K in H_j. Now, $B_j K = K$ and $B_j K^\perp = B_j(\tilde{K} \oplus H_j^\perp) = \tilde{K}$ and these spaces are complementary.

In the other case, when $K \subset H_j^\perp$ then $H_j \subset K^\perp$ so again we have that $B_j K = \{0\}$ and $B_j K^\perp$ are complementary. This completes the proof of the first part of the lemma.

To see the second part, let us first show that we get the decomposition

$$V = (V \cap H_j) \oplus (V \cap H_j^\perp). \tag{20}$$

This follows from the fact from [5] that since the datum is geometric then $(V, V^\perp)$ is a critical pair. Also from there, the proof of Lemma 7.12, we have that

$$\operatorname{tr}_H(B_j P_V) = \dim(B_j V)$$

where P_V is the orthogonal projection onto V and since $B_j P_V$ is a contraction we get that there are linearly independent vectors $\{v_l | l = 1, \ldots, \dim(B_j V)\}$ such that $\|B_j P_V v_l\| = \|P_V v_l\| = \|v_l\|$. The latter of these equalities says that $v_l \in V$ and then the former says that $v_l \in H_j$. Now for any vector $v \in V$ which is orthogonal to each v_l we must have that $B_j P_V v$ is the zero vector so $P_V v = v$ is in $H_j^\perp$. This proves that the decomposition (20) holds.

Let us now assume that $K = \cap_j H_j^a$ and show that

$$V = (V \cap K) \bigoplus \left(\sum_{j=1}^{m} (V \cap H_j^{a\perp}) \right).$$

We see immediately that the space on the right hand side is a subspace of V and that each constituent of the sum is orthogonal to K so the sum is a subspace of $V \cap K^\perp$. What is left to show is that the right hand side contains the whole of V. So, take a vector in V which is orthogonal to each term in the sum on the right. It therefore lies in the orthogonal complement of $V \cap H_j^{a\perp}$ for each j. But from (20) we then get that it is in $V \cap H_j^a$ for each j and thus in $\cap_j (V \cap H_j^a)$ which equals $V \cap K$.

To complete the proof of the lemma we note that if U and V are critical then so are $U \cap V$ and $U + V$. This is a simple consequence of (19) the general relationship between the dimensions of U, V, $U \cap V$ and $U + V$, see also [9]. Thus we arrive at the criticality of $V \cap K$. Then $(V \cap K)^\perp$ is critical since the orthogonal complement of a critical space is critical in the geometric set-up and so again by appealing to the lemma in [9] we get that $V \cap K^\perp = (V \cap K)^\perp \cap V$ is critical. □

Remark 11. This lemma shows that the independent decomposition of H is also a critical decomposition and that any maximal critical decomposition of H is a refinement of the independent one.

We also note that a maximal critical decomposition is not unique. For example for Hölder's inequality, we can take any orthogonal basis $\{e_1, \ldots, e_n\}$ of H and $H = \oplus \langle e_i \rangle$ is a maximal critical decomposition. The independent decomposition here is simply H.

However, some parts of the decomposition are shared between any maximal critical decomposition. For example, in the Loomis–Whitney type situation when we let $\{e_i | i = 1, \ldots, 6\}$ be an orthogonal basis for H and B_j for $j = 1, 2, 3$ be the projection onto the span of e_{2j-1} and e_{2j} then any maximal critical decomposition is a refinement of the decomposition

$$H = \langle e_1, e_2 \rangle \oplus \langle e_3, e_4 \rangle \oplus \langle e_5, e_6 \rangle$$

and this decomposition is the independent decomposition of H.

We also note that even K_{dep} need not have a unique maximal critical decomposition. As an example take the case when $\{e_i | i = 1, \ldots, 4\}$ is a orthogonal basis for H and B_j for $j = 1, 2, 3$ are the orthogonal projections onto $\langle e_1, e_3 \rangle$, $\langle e_2, e_3 \rangle$ and $\langle e_1 + e_2, e_3 + e_4 \rangle$ respectively. Then $K_{\text{dep}} = H$ and

$$H = \langle e_1, e_2 \rangle \oplus \langle e_3, e_4 \rangle = \langle e_1 + e_3, e_2 + e_4 \rangle \oplus \langle e_1 - e_3, e_2 - e_4 \rangle$$

are two maximal critical decompositions of H.

We have from Lemma 7 that since the datum we are working with is geometric then for any critical subspace V, the orthogonal complement $V^\perp$ is also critical. Therefore, from the decomposition (20) we get for any j that

$$H = (V \cap H_j) \oplus (V \cap H_j^\perp) \oplus (V^\perp \cap H_j) \oplus (V^\perp \cap H_j^\perp). \tag{21}$$

This shows that $B_j^* B_j P_V = P_V B_j^* B_j$ and furthermore that

$$B_j P_V = B_j P_V B_j^* B_j. \tag{22}$$

4. Determination of optimisers

With the set-up of previous sections in hand we can state our main theorem as follows.

Theorem 12. *Let $((B_j),(p_j))$ be a geometric Brascamp–Lieb datum as above and let $\oplus_{k=0}^{k_0} K_k \oplus K_{\mathrm{dep}}$ be the independent decomposition of H.*

Assume that (f_j) is an extremiser for this datum.

Then there exist integrable functions $u_k : H_k \to \mathbb{R}$, $k = 1, \dots, k_0$, a critical decomposition $K_{k_0+1} \oplus \cdots \oplus K_{k_1}$ of K_{dep}, positive constants c_j for $j = 1, \dots, m$ and d_k for $k = k_0+1, \dots, k_1$ and an element b from K_{dep} such that

$$f_j(x) = c_j \prod_{k=1}^{k_0} u_k(P_{j,k} B_j^* x) \prod_{k=k_0+1}^{k_1} e^{-d_k \langle P_{j,k} B_j^* x, P_{j,k}(B_j^* x + b)\rangle} \tag{23}$$

where $P_{j,k}$ is the orthogonal projection from H to $H_j \cap K_k$.

Conversely, all functions of this form are optimisers for this problem.

Proof. Take an extremiser (f_j) which consists of Schwartz functions. We will remove this restriction at the end of the proof.

Since the datum $((B_j),(p_j))$ is extremisable it is a theorem from [5] that it is also gaussian-extremisable. Then the whole theory from the previous section applies and since we know that we have equality in (16) for gaussian optimisers we have for (f_j) that

$$\frac{\partial}{\partial t} \int F(x,t) \, \mathrm{d}x = 0$$

for all $t > 0$ where we have continued with the notation introduced in the Section 2. Let us fix a time $t > 0$ and suppress the dependence of the various quantities on it for the time being.

As noted above, the fact that (f_j) is an extremiser means that the quantity (14) must be 0 for all $t > 0$ and all $x \in H$. We must then have $\langle (I-P)A, A\rangle = 0$. We have that $\langle P\alpha, \alpha\rangle \le \langle \alpha, \alpha\rangle$ for all vectors α and there is equality here if and only if $P\alpha = \alpha$, which means that α is in the image of P. We recall that $P = TT^*$ where

$$T = \begin{pmatrix} --- & p_j^{\frac{1}{2}} B_1 & --- \\ & \vdots & \\ --- & p_j^{\frac{1}{2}} B_m & --- \end{pmatrix}.$$

We note that T is a linear transformation from H to $\oplus_j H_j$ and P is the projection onto the span of the column vectors of T. Therefore we see that the quantity (14) is 0 if and only if there exists a map $\beta : H \to H$ such that

$$A(x) = T\beta(x) \tag{24}$$

for almost every $x \in H$.

If we read off the rows in the above equation we find that $p_j^{\frac{1}{2}} \nabla h_j(B_j x) = p_j^{\frac{1}{2}} B_j \beta(x)$ or

$$\nabla h_j(B_j x) = B_j \beta(x). \tag{25}$$

Then we see that

$$\nabla \log F(x) = \sum_{j=1}^{m} p_j B_j^* \nabla h_j(B_j x) = \sum_{j=1}^{m} p_j B_j^* B_j \beta(x) = \beta(x)$$

where we have used the geometricity of the datum for the last equality. We note from this equation that β is smooth and (24) must hold for all x.

By using (25) we can make the following calculation:

$$\langle \nabla \log F(x), b_j \rangle = \langle \beta(x), b_j \rangle = \langle B_j \beta(x), e_j \rangle = \langle \nabla h_j(B_j x), e_j \rangle$$

for any $b_j = B_j^* e_j$ where $e_j \in H_j$. If we differentiate this equality with respect to a vector $b_j^\perp = B_j^{\perp *} e_j^\perp$ where $e_j^\perp \in H_j^\perp$ and $B_j^\perp$ is the orthogonal projection onto $H_j^\perp$ we get that

$$D^2(\log F)(b_j, b_j^\perp) = 0 \tag{26}$$

as the quantity on the far right hand side of the last chain of equalities is constant in the direction of $b_j^\perp$. This means that $\log F$ has the form $\log F = u_j^{\|}(B_j x) + u_j^\perp(B_j^\perp x)$ where $u_j^{\|}$ and $u_j^\perp$ are smooth.

Since we can make this calculation for any j we have established the equalities

$$\log F = u_j^{\|}(B_j x) + u_j^\perp(B_j^\perp x) = u_{j'}^{\|}(B_{j'} x) + u_{j'}^\perp(B_{j'}^\perp x) \tag{27}$$

for all j, j'.

In the following two lemmas we use this equality to determine the optimisers.

Lemma 13. *We can write*

$$\log F = \left(\sum_{k=1}^{k_0} u_{K_k}(P_{K_k} x) \right) + u_{K_{\mathrm{dep}}}(P_{K_{\mathrm{dep}}} x) \tag{28}$$

where

$$H = \left(\bigoplus_{k=1}^{k_0} K_k \right) \bigoplus K_{\mathrm{dep}}$$

is the independent decomposition of H.

Proof. The lemma will follow if we show that the second derivative of $\log F$ with respect to any pair of vectors from different components of this decomposition is identically zero. If the vectors come from two distinct independent subspaces, K_k and $K_{k'}$ say, there must be a j such that $K_k \subset H_j$ and $K_{k'} \subset H_j^\perp$ or the other way around. Then the result follows immediately by taking $b_j \in H_j$ and $b_j^\perp \in H_j^\perp$ in (26).

Let us now turn to the case when one of the vectors, b_1, comes from K_k for some k and the other one, b_2, from K_{dep}. Assume that $K_k = \cap_j H_j^a$ where as before H_j^a is either H_j or $H_j^\perp$. From the definition of K_{dep} we have that

$$K_{\mathrm{dep}} \subset K_k^\perp = \sum_j H_j^{a\perp}$$

so b_2 can be written as a linear combination of vectors in $H_j^{a\perp}$. Since b_1 lies in H_j^a for all j we see that $D^2(\log F)(b_1, b_j^{a\perp}) = 0$ for any $b_j^{a\perp} \in H_j^{a\perp}$. Thus $D^2(\log F)(b_1, b_2) = 0$. □

For the non-independent part we have the following lemma.

Lemma 14. *Assume that H has no independent subspaces. Then any optimiser is a gaussian.*

Proof. Let us take the gradient of (27). This gives

$$\begin{aligned} \nabla \log F(x) &= B_j^* \nabla u_j^{\|}(B_j x) + B_j^{\perp *} \nabla u_j^\perp (B_j^\perp x) \\ &= B_{j'}^* \nabla u_{j'}^{\|}(B_{j'} x) + B_{j'}^{\perp *} \nabla u_{j'}^\perp (B_{j'}^\perp x). \end{aligned} \tag{29}$$

We use the Fourier transform to retrieve information from this equation. Note that

$$\nabla \log F(x) = \sum_{j=1}^m p_j \frac{B_j^* \nabla f_j(B_j x)}{f_j(B_j x)}$$

so from Lemma 5 we see that $\nabla \log F$ has linear growth in x. From (29) it is now evident that $B_j^* \nabla u_j^{\|}(B_j x)$ and $B_j^{\perp *} \nabla u_j^\perp (B_j^\perp x)$ have at most linear growth in x and we are therefore justified to take the Fourier transform of (29).

Let us now note that the Fourier transform of a function w of the form $w(x) = u(B_j x)$ is supported in H_j. To see this, we take a test function ϕ and calculate as follows.

$$\begin{aligned} \int_H \hat{w}(\xi)\phi(\xi)\,\mathrm{d}\xi &= \int_H \int_H u(B_j x)\phi(\xi) e^{-2\pi i \langle x, \xi\rangle}\,\mathrm{d}\xi\,\mathrm{d}x \\ &= \int_{H_j}\int_{H_j^\perp}\int_{H_j}\int_{H_j^\perp} u(x_1)\phi(\xi_1,\xi_2) \\ &\qquad\qquad e^{-2\pi i\langle x_1,\xi_1\rangle} e^{-2\pi i\langle x_2,\xi_2\rangle}\,\mathrm{d}\xi_2\,\mathrm{d}\xi_1\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \int_{H_j}\int_{H_j} u(x_1)\phi(\xi_1, 0) e^{-2\pi i\langle x_1,\xi_1\rangle}\,\mathrm{d}\xi_1\,\mathrm{d}x_1 \end{aligned}$$

since

$$\int_{H_j^\perp}\int_{H_j^\perp} \phi(\xi_1,\xi_2) e^{-2\pi i\langle x_2,\xi_2\rangle}\,\mathrm{d}\xi_2\,\mathrm{d}x_2 = \phi(\xi_1, 0).$$

The last integral in the above expression is clearly 0 if ϕ is supported away from H_j. Now, equation (29) says that $\nabla \log F$ is the sum of a function which depends only on $B_j x$ and another which depends only on $B_j^\perp x$. It is therefore clear that the Fourier transform of $\nabla \log F$ is supported on $H_j \cup H_j^\perp$. Since this holds for any j we get that it is in fact supported on

$$\bigcap_j (H_j \cup H_j^\perp).$$

This intersection contains only the origin by the assumption that H has no independent subspaces.

It is well known that the Fourier transform of a distribution supported at the origin is a polynomial and we have thus established that $\nabla \log F$ is a polynomial and in fact a linear polynomial by the growth estimate above. Equation (25), together with the fact that $\nabla \log F = \beta$, gives that f_j is a gaussian. $\square$

From this lemma it is clear that the function $u_{K_{\text{dep}}}$ appearing in (28) is a gaussian and by the theory established in [5] for gaussian optimisers we know that since the datum is geometric there exists a maximal critical decomposition

$$K_{\text{dep}} = \bigoplus_{k=k_0+1}^{k_1} K_k$$

such that the purely quadratic term in this gaussian is the tensor product of multiples of the identity operator on each K_k appearing in this decomposition. Thus we have shown that with this decomposition we can write

$$\log F(x) = \sum_{k=1}^{k_0} u_{K_k}(P_{K_k} x) - \sum_{k=k_0+1}^{k_1} d_k \langle P_{K_k} x, P_{K_k}(x + b_k)\rangle$$

and so

$$\beta(x) = \nabla(\log F)(x) = \sum_{k=1}^{k_0} P_{K_k}^* \nabla u_{K_k}(P_{K_k} x) - \sum_{k=k_0+1}^{k_1} d_k P_{K_k}^* P_{K_k}(2x + b_k) \tag{30}$$

which gives

$$B_j \beta(x) = \sum_{k=1}^{k_0} B_j P_{K_k}^* \nabla u_{K_k}(P_{K_k} x) - \sum_{k=k_0+1}^{k_1} d_k B_j P_{K_k}^* P_{K_k}(2x + b_k).$$

Now, each term in the first part is zero unless $K_k \subset H_j$ in which case $P_{K_k} = P_{K_k} B_j^* B_j$. For the second part we have from (22) that $B_j P_{K_k}^* P_{K_k} = B_j P_{K_k}^* B_j^* B_j$. Therefore we have shown that $B_j \beta(x)$ depends only on $B_j x$ and moreover we see that $B_j \beta(x) = \nabla h_j(B_j x)$ where

$$h_j(x) = \sum_{\substack{k=1 \\ K_k \subset H_j}}^{k_0} u_{K_k}(P_{K_k} x) - \sum_{k=k_0+1}^{k_1} d_k \langle P_{H_j \cap K_k} x, P_{H_j \cap K_k}(x + b_k)\rangle. \tag{31}$$

This shows that $f_j(\cdot, t)$ must have the prescribed form for any $t > 0$. Since the set of tuples allowed by the theorem is a closed set in $L^1 \times \cdots \times L^1$ we get that $(f_j) = (f_j(\cdot, 0))$ must also have this form as by the theory of the heat equation each f_j is the L^1 limit $\lim_{t \to 0} f_j(\cdot, t)$.

Finally, let us remove the restriction that (f_j) is a tuple of Schwartz functions. Thus let (f_j) be an optimiser where each f_j is only assumed to be an L^1 function on H_j. Consider the tuple $(f_j * g_j)g_j$ where $g_j(x) = e^{-\|x\|^2_{H_j}}$. Since the datum we are working with is geometric then (g_j) is an optimiser and $((f_j * g_j)g_j)$ is an optimiser as well, see Lemma 6.3 in [5]. Each function in this new tuple is a positive Schwartz function so by what we have proved it is clear that this optimiser has the form (23). Now note that if $(f * g)g = k$ where $k(x) = k_P(Px)k_{P^\perp}(P^\perp x)$, P and $P^\perp$ are projections onto orthogonal subspaces, and $g(x) = g_P(Px)g_{P^\perp}(P^\perp x)$ is strictly positive then also $f(x) = f_P(Px)f_{P^\perp}(P^\perp x)$. Furthermore, if g and h are gaussians, then f is also a gaussian. This shows that the results of Lemmas 13 and 14 hold for the L^1 tuple (f_j) and this completes the demonstration that any tuple of optimisers must have the prescribed form (23).

To prove the converse, that all tuples of the form (23) are optimisers, we first of all make the following remark. Assume that V is a critical subspace. Then from (21) we can write $H_j = (H_j \cap V) \oplus (H_j \cap V^\perp)$. Assume further that each f_j has the form $f_j(x) = f_{jV}(P_V B_j^* x) f_{jV^\perp}(P_{V^\perp} B_j^* x)$. Then

$$\int_H \prod_{j=1}^m f_j^{p_j}(B_j x)\,\mathrm{d}x = \int_V \prod_{j=1}^m f_{jV}^{p_j}(B_j|_V x_1)\,\mathrm{d}x_1 \int_{V^\perp} \prod_{j=1}^m f_{jV^\perp}^{p_j}(B_j|_{V^\perp} x_2)\,\mathrm{d}x_2$$

so (f_j) is an optimiser if (f_{jV}) and $(f_{jV^\perp})$ are optimisers for the data $((B_j|_V),(p_j))$ and $((B_j|_{V^\perp}),(p_j))$ respectively.

By repeating this splitting we may thus reduce to showing the following two things. Firstly, we must show that (f_j) with $f_j = c_j g$ where c_j is a constant and g is an integrable function is an optimiser in the case when $H_j = H$ for all j, $B_j = \mathrm{id}_H$, and the Brascamp–Lieb inequality reduces to

$$\int_H \prod_j f_j^{p_j}(x)\,\mathrm{d}x \le \prod_j \left(\int_H f_j(x)\,\mathrm{d}x\right)^{p_j}$$

with $\sum_j p_j = 1$. The proof is immediate by writing both sides of the inequality in terms of g.

Secondly, we must show that the gaussian tuple $(f_j) = (e^{-d\langle x, x+B_j b\rangle})$ with $d > 0$ and $b \in H$ is an optimiser for the Brascamp–Lieb inequality in the case when H has no independent subspaces and no proper critical subspace. However, even without these restrictions and only with the condition that $((B_j),(p_j))$ is geometric, it is well known that this tuple is an optimiser. $\square$

Finally, let us drop the condition that $((B_j),(p_j))$ is geometric. From Theorem 7.13 in [5] we have that any extremisable datum is equivalent to a geometric datum. More specifically, the equations

$$M = \sum_{j=1}^m p_j B_j^* S_j B_j \tag{32}$$

$$S_j^{-1} = B_j M^{-1} B_j^* \quad j = 1, \dots, m. \tag{33}$$

have a solution M and S_j with symmetric positive definite linear transformations and $((B'_j),(p_j))$ with $B'_j = S_j^{\frac{1}{2}} B_j M^{-\frac{1}{2}}$ is a geometric datum. Also, if (f_j) is an optimiser for $((B_j),(p_j))$ then $(f_j \circ S_j^{-\frac{1}{2}})$ is an optimiser for $((B'_j),(p_j))$ and conversely, if (f'_j) is an optimiser for $((B'_j),(p_j))$ then $(f'_j \circ S_j^{\frac{1}{2}})$ is an optimiser for $((B_j),(p_j))$.

As a direct consequence of this and Theorem 12 we get the following.

Theorem 15. *Let $((B_j),(p_j))$ be an extremisable Brascamp–Lieb datum.*

Assume that (f_j) is an extremiser for this datum.

Let $((B'_j),(p_j))$ be the geometric datum equivalent to $((B_j),(p_j))$ and let M and S_j be such that $B'_j = S_j^{\frac{1}{2}} B_j M^{-\frac{1}{2}}$. Furthermore, let $\oplus_{k=1}^{k_0} K_k \oplus K_{\mathrm{dep}}$ be the independent decomposition of H corresponding to the datum $((B'_j),(p_j))$. Then there exist integrable functions $u_k : H_k \to \mathbb{R}$, $k = 1,\ldots,k_0$, a critical decomposition $K_{k_0+1} \oplus \cdots \oplus K_{k_1}$ of K_{dep}, positive constants c_j for $j = 1,\ldots,m$ and d_k for $k = k_0+1,\ldots,k_1$ and an element b from K_{dep} such that

$$f_j(x) = c_j \prod_{k=1}^{k_0} u_k(P_{j,k} B_j'^{*} S_j^{\frac{1}{2}} x) \prod_{k=k_0+1}^{k_1} e^{-d_k \langle P_{j,k} B_j'^{*} S_j^{\frac{1}{2}} x, P_{j,k}(B_j'^{*} S_j^{\frac{1}{2}} x + b)\rangle} \tag{34}$$

where $P_{j,k}$ is the orthogonal projection from H to $H_j \cap K_k$.

Conversely, all functions of this form are optimisers for this problem.

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