A World in Fragments

Studies on the Encyclopedic Manuscript GKS 1812 4to

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Algorismus: Hindu-Arabic Arithmetic in GKS 1812 4to

Kristín Bjarnadóttir
and Bjarni V. Halldórsson

The Old Norse treatise *Algorismus* is a prose translation of the Latin hexameter poem *Carmen de Algorismo*, written in France in the early thirteenth century by Alexander de Villa Dei (ca. 1170–1240), who was a canon at St. Andrew's Cathedral in Avranches.[1] The Old Norse translation exists in four manuscripts, one of which is GKS 1812 4to, preserved at the Árni Magnússon Institute for Icelandic Studies in Reykjavík. The other manuscripts are AM 544 4to, AM 685 d 4to, and AM 736 III 4to, preserved in the Arnamagnaean Institute in Copenhagen. The text of AM 544 4to was published in 1892–1896 by Professor Finnur Jónsson, who suggested that *Carmen* had been translated into Old Norse before 1270.[2] Helgi Guðmundsson deems it possible that the translation existed in Viðey Abbey in the early fourteenth century.[3]

1 Barnabas Hughes, "Franciscans and Mathematics II," *Archivum franciscanum historicum* 77 (1984): 8–9.

2 *Hauksbók. Udgiven efter de arnamagnæanske Håndskrifter no. 371, 544 og 675, 4° samt forskellige Papirhåndskrifter*, ed. Eiríkur Jónsson and Finnur Jónsson (Copenhagen: Det Kongelige Nordiske Oldskrift-Selskab, 1892–1896), 417–424; cxxxii. Mathematician Otto B. Bekken translated *Algorismus* into modern Norwegian in 1985 and explained its text in cooperation with linguist Marit Christoffersen, see Otto Bekken and Marit Christoffersen, *Algorismus i Hauksbok*, Skrifter 1985:1 (Kristiansand: Agder Distrikthøgskole, 1985). One of the present authors has published various studies of the *Algorismus*, see Kristín Bjarnadóttir, "Algorismus," in *Study the Masters: The Abel-Fauvel Conference. Gimlekollen Mediacentre – Kristianssand, June 12–15, 2002*, ed. Otto B. Bekken and Reidar Mosvold (Gothenburg: Nationellt Centrum för Matematikutbildning, NCM, 2003), 99–108; Kristín Bjarnadóttir, *Mathematical Education in Iceland in Historical Context. Socio-Economic Demands and Influences*. IMFUFA tekst (Roskilde: Roskilde University electronic library, 2007), 43–46; Kristín Bjarnadóttir, "Algorismus: Fornt stærðfræðirit í íslenskum handritum," *Netla, veftímarit um uppeldi og menntun* (2004).

3 Helgi Guðmundsson, *Um Kjalnesingasögu: nokkrar athuganir*, Studia Islandica 26 (Reykjavík: Menningarsjóður, 1967), 68.

The Old Norse *Algorismus* contains an explanation of the Hindu-Arabic number notation, including the decimal place-value notation and calculation methods in seven algorithms: addition, subtraction, doubling, halving, multiplication, division, and extraction of roots, which is further subdivided into the square root and the cubic root. These methods have been relayed to *Algorismus* via *Carmen de Algorismo,* which is a versified explanation of Hindu-Arabic arithmetic, built on the work by Muḥammad Ibn Mūsā al-Khwārizmī (ca. 780–850), most likely on *Liber Alchorismi,* one of the twelfth-century Latin elaborations of his book *On the Indian Calculation,* the Arabic version of which has been lost. This conjecture is based on the order of the arithmetic operations, which varies in the different early works written under the influence of his treatise (*Dixit Algorizmi*, *Liber Ysagogarum Alchorismi*, *Liber Alchorismi*, *Liber Pulueris*).[4] The cubic root, contained in the *Algorismus,* is not included in these works and must have been acquired from another source.

In this article, the focus is on *Algorismus* in the manuscript GKS 1812 4to. First, we explain the arithmetic operations in the poem and the treatise, then we explore incidences where *Algorismus* deviates from *Carmen de Algorismo*, and discuss the passage in *Algorismus* on numbers related to the elements. After a brief account of the four extant manuscripts of *Algorismus,* we compare the copies of *Algorismus* in GKS 1812 4to, AM 544 4to, AM 685 d 4to, and AM 736 III 4to.

7.1 Arithmetic Operations in *Carmen de Algorismo* and *Algorismus*

The *Carmen de Algorismo* exists in a great number of manuscripts, preserved in many libraries, for example in France, Great Britain, and the Netherlands. It is considered to have played an even greater role in distributing Hindu-Arabic number notation in Northern Europe than the well-known *Liber abaci* by Leonardo of Pisa.[5] One feature to be mentioned is that both the Old Norse manuscript AM 544 4to, and *Carmen de Algorismo* in the Latin MS. Auct. F.5.29, preserved in the Bodleian Library in Oxford and dated to the thirteenth century, have chapter headings which are found

4 André Allard, "Les algorismes et leurs auteurs," *Le Calcul Indien,* ed. André Allard (Paris: Blanchard; Namur: Societé des Études Classiques, 1992), xxxi. The book was translated into Latin under the title *De numero Indorum.* The lost Arabic original is usually referred to as *Kitāb fī l-ḥisāb al-hindī.* See also Chapter 6 by Abdelmalek Bouzari in this volume.

5 Jacqueline Stedall, personal communication, 2008.

neither in the two known printed versions of *Carmen de Algorismo* nor in the other manuscripts of *Algorismus*.[6]

Carmen de Algorismo is in hexameter verse and is meant to be recited and memorized. The beginning of the poem reads as follows: [7]

> *Hec algorismus ars presens dicitur; in qua*
> *Talibus Indorum fruimur bis quinque figuris*
> *0 · 9 · 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1 ·*
> *Prima significat unum : duo vero secunda :*
> *Tercia significat tria : sic procede sinistre*
> *Donec ad extremam venies, qua cifra vocatur*.[8]

The ten digits in the third line of the poem are the only incidence where the new Hindu-Arabic numerals are presented in *Carmen*, see Figure 7.1. Everywhere else numbers are expressed in words. *Carmen de Algorismo* is believed to be the first work in Latin where the zero, *cifra*, is presented as a digit.[9]

The poem explains algorithms which are now familiar to most of us but it does not give any examples. It is not known how the poem was used as an aid to computation, but one may assume that calculations were made on tablets or a flat surface, strewn with sand, or else on a wax tablet.

The initial text in *Algorismus* is a direct translation of the Latin original in *Carmen*:

> List þessi heitir Algorismus. Hana fundu fyrst indverskir menn með tíu stöfum er svo eru ritaðir 0 · 9 ·8 · 7 · 6 · 5 · 4 · 3 · 2 · 1. Hinn fyrsti stafur merkir einn í fyrsta stað. En annar tvo. En þriðji þrjá. Og hver eftir því sem skipaður er allt til hins síðasta er cifra heitir.[10]

6 "Carmen de Algorismo," in *The Earliest Arithmetics in English*, ed. Robert Steele (Millwood, N.Y.: Kraus Reprint, 1988), 72–80; "Carmen de Algorismo," in *Rara mathematica*, ed. James Orchard Halliwell (London: Samuel Maynard, 1841), 73–83.

7 In the following examples, the Latin texts are from "Carmen de Algorismo," printed in *The Earliest Arithmetics in English*. The Old Norse text is from the manuscript GKS 1812 4to, with modern Icelandic orthography. Translations into English are by Kristín Bjarnadóttir.

8 "Carmen de Algorismo," *The Earliest Arithmetics in English*, 72.

9 Suzan Rose Benedict, "A Comparative Study of the Early Treatises Introducing into Europe the Hindu Art of Reckoning," PhD thesis (University of Michigan, 1913), 123; Guy Beaujouan, "D'Alexandre de Villedieu à Sacrobosco," *Homenaje à Millás-Vallicrosa* (Barcelona: Consejo Superior de Investigaciones Científicas, 1954), 1:106.

10 GKS 1812 4to, 13v.

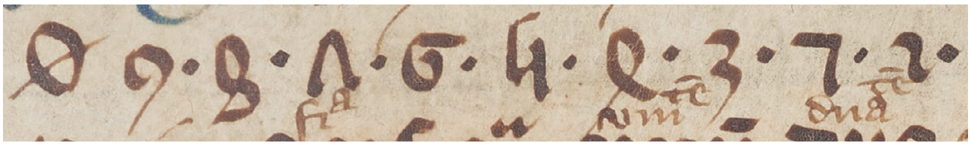

FIGURE 7.1 – The ten digits of Hindu-Arabic numerals in *Carmen de Algorismo* in MS. Auct. F.5.29 (Bodleian Library, University of Oxford 2021).

> (This art we call Algorismus. It was first found by Indians and arranged by ten digits which are written thus: 0 · 9 ·8 · 7 · 6 · 5 · 4 · 3 · 2 · 1. The first digit signifies one in the first place. Two the second. The third three. And each as it is ordered until the last one which is called cifra [the zero].)

Natanael Beckman claimed that the typeface of the numerals in the manuscripts of *Algorismus* proceeds from Johannes de Sacrobosco and that his treatise, called *Algorismus vulgaris*, was known in Iceland around 1280.[11] Finnur Jónsson also remarked that the writing style of the numerals in the Old Norse *Algorismus* as presented in Figure 7.2 belongs to a tradition of the thirteenth century up to 1270. Certainly, that did not prove that the translation is older than from about 1270, but Jónsson considered it plausible.[12] This conjecture is further supported by a comparison to the numerals in MS. Auct. F.5.29, dated to the thirteenth century, see Figure 7.1.

There are more connections to Sacrobosco and his *Algorismus vulgaris* of about 1225 which is a longer text than *Carmen*. Sacrobosco's text cites three verses from *Carmen*, for example on the operations:

Carmen and *Algorismus vulgaris:*

> *Subtrahis aut addis a dextris vel mediabis;*
> *A leua dupla, diuide, multiplicaque;*
> *Extrahe radicem semper sub parte sinistra*.[13]
>
> (Subtract or add from the right, or halving;
> from left double, divide, multiply;
> extract roots always from the left side.)

11 *Alfræði íslenzk*, vol. 2, *Rímtǫl*, ed. N. Beckman and Kr. Kålund (Copenhagen: Samfund til Udgivelse af gammel nordisk Litteratur, 1915), xxxviii.

12 *Hauksbók*, ed. Eiríkur Jónsson and Finnur Jónsson, cxxxii.

13 "Carmen de Algorismo," *The Earliest Arithmetics in English*, 73; *Petri Philomeni de Dacia in Algorismum vulgarem Johannis de Sacrobosco commentarius*, ed. Maximilianus Curtze (Copenhagen: A.F. Høst & fil., 1897), 7.

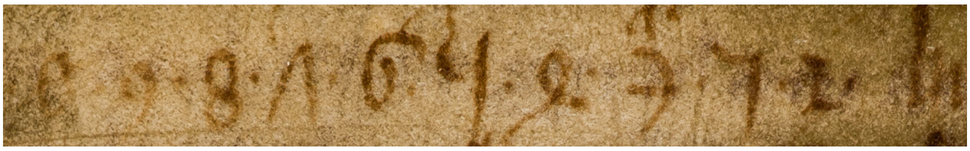

FIGURE 7.2 – The ten digits of Hindu-Arabic numerals in *Algorismus* in GKS 1812 4to. Photo: Sigurður Stefán Jónsson.

Algorismus in GKS 1812 4to:

> Frá hinni hægri hendi skalt þú af taka og við leggja og skipta í helminga en frá vinstri hendi skalt þú tvöfalda og skipta og margfalda og svo draga rót undan hvorutveggju.[14]
>
> (From the right hand you shall deduct and add and split in halves but from the left hand you shall double and divide and multiply and also extract roots from under both.)

The arithmetic operations of addition, subtraction and division, explained in *Carmen* and *Algorismus*, are largely similar to present methods used in paper-and-pencil arithmetic.

Multiplying two composite numbers, however, proceeds from the left, as opposed to common modern algorithms. The numbers to be multiplied are arranged so that the digit farthest to the right of the multiplicand is placed below the first digit (from the left) of the multiplier. The multiplicand is multiplied by this digit which then disappears under the product. Then the multiplicand is moved one place to the right so that the rightmost digit is placed below the second digit of the multiplier in the upper row and the lower number is now multiplied by this in the same manner. The product is written over the multiplying digit and added to the next digits to the left.

Carmen and *Algorismus* do not illustrate their algorithms on the four arithmetic operations by examples. The following example is constructed for clarification in this chapter:

Multiply 523 by 217:
First 523 is multiplied by 2 and 2 disappears under the product:

217	**1046**17	10**9837**
523	523	[523 is now multiplied by 1, etc.]

14 GKS 1812 4to, 13v.

The first digit on the right of the lower number, 3, is moved one place to the right below the second digit of the upper number, 1, which now is the multiplying digit. This procedure continues until all digits of the upper number have been used as multipliers. The advantage of multiplying from the left is that the product of the digits are added to the previous product step by step.

Doubling and halving were treated as distinct algorithms. Both operations are known from antiquity, even as replacements for multiplication and division.[15] Doubling was done from the left as is customary in mental arithmetic, while halving was done from the right.

The square root was drawn from the left. The method is not much different from what was customary before calculators became a common possession of school children. Extracting cubic root also had separate sections in *Carmen*, *Algorismus* and *Algorismus vulgaris*. The section in *Carmen* is considered to be the first incidence where extracting cubic root is introduced in Latin.[16] It is not contained in the Latin translations of al-Khwārizmī's work although it is known from the work *Āryabhaṭīya* by the Indian mathematician Āryabhaṭa (b. 476).[17]

7.2 Deviations of *Algorismus* from *Carmen de Algorismo*

The poem *Carmen de Algorismo* describes the Hindu-Arabic number notation in general terms only. The treatise *Algorismus* enhances *Carmen* by demonstrating the new system's notation. It extends the first chapters, suggesting a need to clarify the text by numerical examples, while several repetitions in *Carmen* are omitted in the translation. Immediately after *Carmen's* definition of the decimal place value notation, the Old Norse translation inserts examples, shown here within square brackets.

> *Ergo, proposito numero tibi scribere, primo*
> *Respicias quis sit numerus; quia si digitus sit,*
> *Una figura satis sibi; sed si compositus sit,*
> *Primo scribe loco digitum post articulum fac*
> *Articulus si sit, cifram post articulum sit.*[18]

15 David Seppala-Holzman, "Ancient Egyptians, Russian Peasants Foretell the Digital Age," *Mathematics Teacher* 100, no. 9 (2007): 632–635.

16 Beaujouan, "D'Alexandre de Villedieu à Sacrobosco," 1:106.

17 Victor J. Katz, *A History of Mathematics: An Introduction* (New York: Harper Collins, 1993), 202; cf. Kim Plofker, *Mathematics in India* (Princeton: Princeton University Press, 2009).

18 "Carmen de Algorismo," *The Earliest Arithmetics in English*, 72.

> Ef þú vilt rita nokkra tölu þá hygg þú að ef það er fingur og rita í fyrsta stað eina hverja fígúru slíka sem þarf [á þessa leið, 8]. En ef þú vilt lið rita þá settu cifru fyrir fígúru [á þessa lund, 70]. Vilt þú samsetta tölu rita þá settu fígúru fyrir lið [sem hér, 65].[19]

> (If you want to write some number then think if it is a digit and write in the first place each figure such as is needed [in this way, 8]. But if you want to write tens then put zero in front of the figure [in this way, 70]. If you want to write a composite number, put a figure in front of the tens [as here, 65].)

Notice that the order of sentences in *Algorismus* is different from the Latin version.

Next, even and odd numbers are presented, where the following addition is inserted in *Algorismus*:

> *Quolibet in numero, si par sit prima figura,*
> *Par erit et totum, quicquid sibi continetur;*
> *Impar si fuerit, totum sibi fiet et impar.*[20]

> Hverja tölu er þú ritar þá er hún jöfn ef [tigum gegnir eða] jafn fingur er umfram. En öll tala er ójöfn ef ójafn fingur er umfram. [Jafnir fingur eru fjórir: 2, 4, 6, 8. En ójafnir aðrir fjórir: 3, 5, 7, 9. En einn er hvorki því að hann er eigi tala heldur upphaf allrar tölu.][21]

> (Each number that you write, then it is even if [it is a multiple of ten or] an even digit is extra; but the whole number is uneven if an uneven digit is extra. [Even digits are four, 2, 4, 6 and 8, and uneven another four, 3, 5, 7, 9. But one is neither as it is not a number but the origin of number.])

The digits inserted are written in Hindu-Arabic mode in all extant *Algorismus* manuscripts. *Algorismus* also inserts a note that the number "one" is neither an even nor an odd number as it is the origin of all numbers. Bekken and Christoffersen have pointed out a likeness to the statement that one is not a number in al-Khwārizmī's *Liber alghoarismi de practica arismetrice* (arithmetic), which again refers to another book on arithmetic, most likely either Euclid's *Elements*, book VII, or *Introduction to Arithmetic*

19 GKS 1812 4to, 13v.
20 "Carmen de Algorismo," *The Earliest Arithmetics in English*, 73.
21 GKS 1812 4to, 13v.

by the Neo-Pythagorean Nicomachus.[22] The citation referred to is the following from the elaboration *Dixit Algorizmi* of al-Khwārizmī's work:

> Et iam patefeci in libro algebr et almucabalah, idest restaurationis et oppositionis, quod uniuersus numerus sit compositus et quod uniuersus numerus componatur super unum. Unum ergo inuenitur in uniuerso numero. Et hoc est quod in alio libro arithmetice dicitur quia unum est radix uniuersi numeri et est extra numerum.[23]
>
> (And I have already explained in the book on algebra and almucabalah, which is on restoring and comparing, that every number is composite and every number is composed of the unit. The unit is therefore to be found in every number. And this is what is said in another book on arithmetic that the unit is the origin of all numbers and is outside numbers.)[24]

The next example inserted into *Algorismus* is when *Carmen*'s text states that there are seven operations: addition, subtraction, doubling, halving, multiplication, division, and root extraction:

> *Septem sunt partes, non plures, istius artis;*
> *Addere, subtrahere, duplare, dimidiare;*
> *Sexta est diuidere, set quinta est multiplicare;*
> *Radicem extrahere pars septima dicitur esse*.[25]

Then *Algorismus* adds that root extraction has two branches, extracting square root and cubic root:

> Í sjö staði er skipt greinum þessarar listar. Heitir hin fyrsta viðurlagning. Önnur afdráttur. Þriðja tvefaldan. Fjórða helminga skipti. Fimmta margfaldan. Sjötta skiptingin. Sjöunda að taka rót undan [og er sú með tveimur greinum. Önnur er að taka rót undan ferskeyttri tölu. En önnur grein er það að draga rót undan átthyrndri tölu þeirri er verpils vöxt hefur.][26]

22 Bekken and Christoffersen, *Algorismus i Hauksbok*, 27.

23 Muḥammad Ibn Mūsā al-Khwārizmī, *Le Calcul Indien*, ed. A. Allard (Paris: Blanchard; Namur: Societé des Études Classiques, 1992), 1.

24 Al-Khwārizmī, *Le Calcul Indien*, 1. English translation by the author, K.B., after André Allard's translation from Latin to French.

25 "Carmen de Algorismo," *The Earliest Arithmetics in English*, 73.

26 GKS 1812 4to, 13v.

(Into seven parts are divided this art's branches. The first one is named addition. Second subtraction. Third doubling. Fourth halves splitting. Fifth multiplication. Sixth the division. The seventh to take root from under [and that one is in two branches. One is to take a root from under a square number. But another branch is extracting a root from under an eight-vertex number, the one which has cubical shape.])

Sacrobosco's elaboration of al-Khwārizmī's work, *Algorismus vulgaris*, states: "*Radicum extractio, et haec dupliciter, quoniam in numeris quadratis et cubicis*" (extraction of roots, which is twofold, since [it applies] to square numbers and cube numbers).[27] This quotation suggests that the translator may have known Sacrobosco's text in addition to Villa Dei's *Carmen*. Sacrobosco claims, however, that there are nine arithmetic operations, counting numeration and progression as operations number one and eight.[28]

Each arithmetic operation is explained in a separate chapter. To multiply, the reader is instructed to arrange the two numbers to be multiplied in columns such that the first digit (from the right) of the multiplier is placed below the last digit of the multiplicand, as explained earlier. However, first one must find the difference between the larger multiplicand and ten, and then subtract the smaller multiplicand from its tens the same number of times as indicated by that difference:

In digitum cures digitum si ducere, major
Per quantes distat a denis respice, debes
Namque suo decuplo tociens delere minorem.[29]

Þar næst skalt þú hugsa hversu mikið hina meiri fígúru skortir á tíu þá er þú vilt margfalda. Og svo margar einingar sem á skortir á tíu svo oft skalt þú hina minni töluna þá er þú vilt margfalda taka af tigum hennar.[30]

(Next you are to think how much the larger figure differs from ten, the one you want to multiply. And so many units as differ from ten so often you are to take the lesser number, the one you want to multiply, from its tens.)

27 *Petri Philomeni de Dacia commentarius*, ed. Curtze, 1.
28 *Petri Philomeni de Dacia commentarius*, ed. Curtze, 1.
29 "Carmen de Algorismo," *The Earliest Arithmetics in English*, 75.
30 GKS 1812 4to, 14v.

Algorismus adds this explanation as the last clarification example:

> Og að þú skiljir þetta margfalda sjö og níu. Níu skortir einn á tíu, því tak þú eina sjö af sjötigum. Þá verða eftir þrír og sextigir það eru sjö sinnum níu. Að slíku skapi mátt þú aðrar tölur reyna.[31]
>
> (So that you understand this, multiply seven and nine. Nine differs by one from ten, therefore, take one seven from seven tens. Then remain three and six tens, that is seven times nine. In that way you may try with other numbers.)

In modern notation this may be written

$$7\times9 = 10\times7 - 1\times7$$

or generalized:

$$a\times b = 10a - (10 - b)a \quad (0 < a, b < 10).$$

Two conclusions may be drawn from this explanation. First, the Latin text's presentation is not considered to be sufficiently clear, so an example is needed. Second, the example demonstrates that the transcriber of GKS 1812 4to is not confident with Hindu-Arabic digits, so he uses words. The use of numerals does not have consistent representation across the manuscripts. In this particular case, the manuscript GKS 1812 4to uses words, AM 544 4to a mixture of words and Roman numerals, and the youngest manuscript containing the whole treatise *Algorismus*, AM 685 d 4to, a mixture of words, Roman numerals, and Hindu-Arabic numerals.

7.3 *Algorismus* on Numbers Related to the Elements

In addition to the numerical samples, a separate chapter is added to the translation. It is on the cubic numbers 8 and 27, their intermediate numbers 12 and 18, and their relation to the elements: earth, water, air, and fire. This chapter does not exist in *Carmen*, and its content is unrelated to the main bulk of *Algorismus* in modern understanding. Its introduction says:

> Hver ferskeytt tala hefur tvær mælingar, það er breidd og lengd. En cubicus tala hefur þrenna mæling. Það er breidd og lengd og þykkt eður hæð. Og því kalla spekingar hvern sýnilegan líkama með þessi tölu saman settan að hann hefur jafnan þessa mæling þrenna. Með því að eilíf speki og einn guð vildi heiminn sýnilegan og líkamlegan

31 GKS 1812 4to, 14v.

skapa, þá setti hann fyrst tvær hinar ystu höfuðskepnur eld og jörð. Því að ekki má náttúrlega sýnilegt vera utan þær. Þar sem eldur gerir ljós og hræring. En jörð staðfesti og hald. En með því að þau hafa þrenna ójafna hvíligleika og gagnstaðlega[32] þá var náttúruleg nauðsyn að setja nokkuð milli þeirra það er samþykkti þeirra ósætti. Og sem fyrr er sagt að eldur og jörð og það allt sem líkamlegt er er með þrefaldri tölu er vér köllum cubicum saman sett þá ritum vér þessa tvo cubus. Ritum vér jörðina þessa leið. Tvisvar sinnum tveir tvisvar, 2, 4, 8. En eldinn svo: þrisvar þrír þrisvar, 3, 9, 27.[33]

(Every quadratic number has two measures, that is breadth and length. But a cubic number has three measures. That is breadth, length, and thickness or height. And therefore wise men say that each visible body is composed out of this number, because it always has these three measures [i.e. dimensions]. As eternal wisdom and one God wanted to create the world visible and corporeal, he first set the two outmost elements, fire and earth. Because nothing can be naturally visible without them. As fire makes light and motion, but earth solidity and hold. But as they have three different and contrary sets of qualities, then there was a natural necessity to add something in between them that would reconcile their discord. And as said previously that fire and earth and everything that is corporeal is combined by a triple number which we call a cube, then we write these two cubes. We write the earth in this way. Twice two twice, 2, 4, 8. But the fire so: thrice three thrice, 3, 9, 27.)

Thus earth was assigned the numerical value $2{\cdot}2{\cdot}2 = 8$ and fire $3{\cdot}3{\cdot}3 = 27$. The text continues, saying that no single mediator existed between these two cubes, two proportional numbers were found by taking the square (4) of the root 2 of the smaller cube, multiplied by the root 3 of the larger cube (4×3) : 2, 4, 12. In the same way, the root of the smaller cube (2) was multiplied by the square (9) of the larger cube (2×9) : 3, 9, 18. These two numbers (12 and 18) belong equally to the two previously mentioned cubes (8 and 27), as 27 contains 18 and half of 18, and 18 contains 12 and half of 12, also 12 contains 8 and half of 8. Similarly, God arranged two elements between fire and earth: air and water. Water contains two attributes and two numbers from earth and one attribute and one number from fire. Air

32 Corrected from "gang staðlega" in GKS 1812 4to. The manuscripts AM 544 4to and AM 685 d 4to have "gagn staðlega" ("opposite", "contrary").

33 GKS 1812 4to, 16r–16v.

contains two attributes from fire and two numbers, but one from earth and one number.

The four elements are thus assigned numerical values: earth, $2^3 = 8$; water, $2^2 \times 3 = 12$; air, $2 \times 3^2 = 18$; fire, $3^3 = 27$. This procedure puts the elements in the correct order by lightness: fire (27), air (18), water (12) and earth (8). These numbers constitute a sesquialterate progression 8:12::12:18::18:27, or in general terms: $n : (n+\frac{1}{2}n)$. The text of *Algorismus* concludes by saying that this can be more perfectly understood from a figure later in the manuscript, called *Cubus Perfectus*.[34]

The notion of the elements has a strong relation to Plato's *Timaeus,* paragraphs 31b–32c. The Roman writer Calcidius had translated the first part (to 53c) of *Timaeus* from Greek into Latin around the year 321 CE.

> And since it [the world] was rightly to be corporeal, visible, and tangible, and there is no perception of anything visible in the absence of fire, or of anything tangible in the absence of solidity, and no solidity without earth, god laid down fire and earth as the foundations of the world body. And since no two things cohere firmly and indissolubly without the binding force of a third … if the body of the world were required to have only length and width but no solidity and were of the same sort as the surface of fully formed bodies, then one mean would suffice [32b] for the cohesion of it and its extreme parts. But as it is, since the world body required solidity, and the cohesion of solids involves never one but two means, the craftsman of the world accordingly inserted air and water between fire and earth, salubriously balancing the same elements so that the relationship between air and water would be the same as that between fire and air, and conversely, so that the relationship in the binding of water and earth vould be the same as that between air and water. And so from the four material elements here named [32c] he fabricated this splendid engine as visible, tangible and bound together by a harmonious proportion in the equilibrium of its parts …[35]

Comparing texts, originally written in different languages and brought together through translations from language to language, is intriguing, but needs vigilance. The texts about the four elements in *Algorismus* and

34 GKS 1812 4to, 16v.

35 Calcidius, *On Plato's* Timaeus, ed. and trans. John Magee (Cambridge, MA: Harvard University Press, 2016), 49–51.

Timaeus bear a remarkable resemblance to each other. Here, the term *staðfesti* in *Algorismus* has been translated by "solidity," the term used in the translation of *Timaeus*, referring to the fact that the earth is a solid and firm body.

Calcidius provided an extensive commentary to his translation of *Timaeus*. In the commentary on the passages above, Calcidius discussed the analogy of the relations between the four elements to continuous proportions, where the air would have "two powers of fire, its fineness and mobility, and one of earth, i.e. its compactness." Similarly, water would have "two powers of earth, i.e. its compactness and corporeality, and one of fire, i.e. its movement, and the substance of water would emerge, that being a body compact, corporeal and mobile." Thus a binding continuity of the world would be given.[36] Calcidius discussed continuous proportions in relation to these items. However, he did not bring up the sequence 8 – 12 – 18 – 27.[37]

The manuscript AM 736 III 4to contains only a fragment of the treatise *Algorismus*. It does not contain the text on the elements and their associated numbers. However, on a different leaf in the same manuscript, a diagram of the four elements is found together with their names and the texts "*bis bini tres xii*" (twice two three 12) associated with water, "*tres trium bis xviii*" (three thrice twice 18) with air, and "*tres trium tres*" (three thrice three) with fire, see Figure 7.3.

The roundels to the right distribute three pairs of qualities: acuity (*acutus* above, *obtusus* below), density (*subtilis* above, *corpulentus* below), and capacity for motion (*mobilis* above, *immobilis* below). Water is associated with three qualities, being corporeal, soft, and mobile. Similarly air is soft, mobile, and light. It seems fair to conclude that we have here the *Cubus Perfectus* which is mentioned in the three complete copies of *Algorismus*.

Diagrams of the elements and the four numbers exist in some British and European manuscripts on medieval cosmology, but those manuscripts are not related to *Algorismus*. For instance, the same sequence of proportions, 27, 18, 12, and 8, and elements, *ignis*, *air*, *aqua*, and *terra*, and the same qualities in roundels on the right appear in St. John's College MS 17 (Oxford Digital Library).[38] A similar schema exists in an eleventh-century manuscript of Boethius, Madrid Biblioteca nacional Vit. 20 fol. 54v.[39] It

36 *On Plato's* Timaeus, 153–155.

37 *On Plato's* Timaeus, 139–145.

38 *The Calendar and the Cloister: Oxford, St John's College MS17* (McGill University Library, Digital Collections Program, 2007), commentary at http://digital.library.mcgill.ca/ms-17.

39 Otto Bekken, *On the Cubus Perfectus of the Algorismus in Hauksbok,* Skrifter 1986:2 (Kristiansand: Agder Distrikthøgskole, fagseksjon for matematikk, 1986), 16.

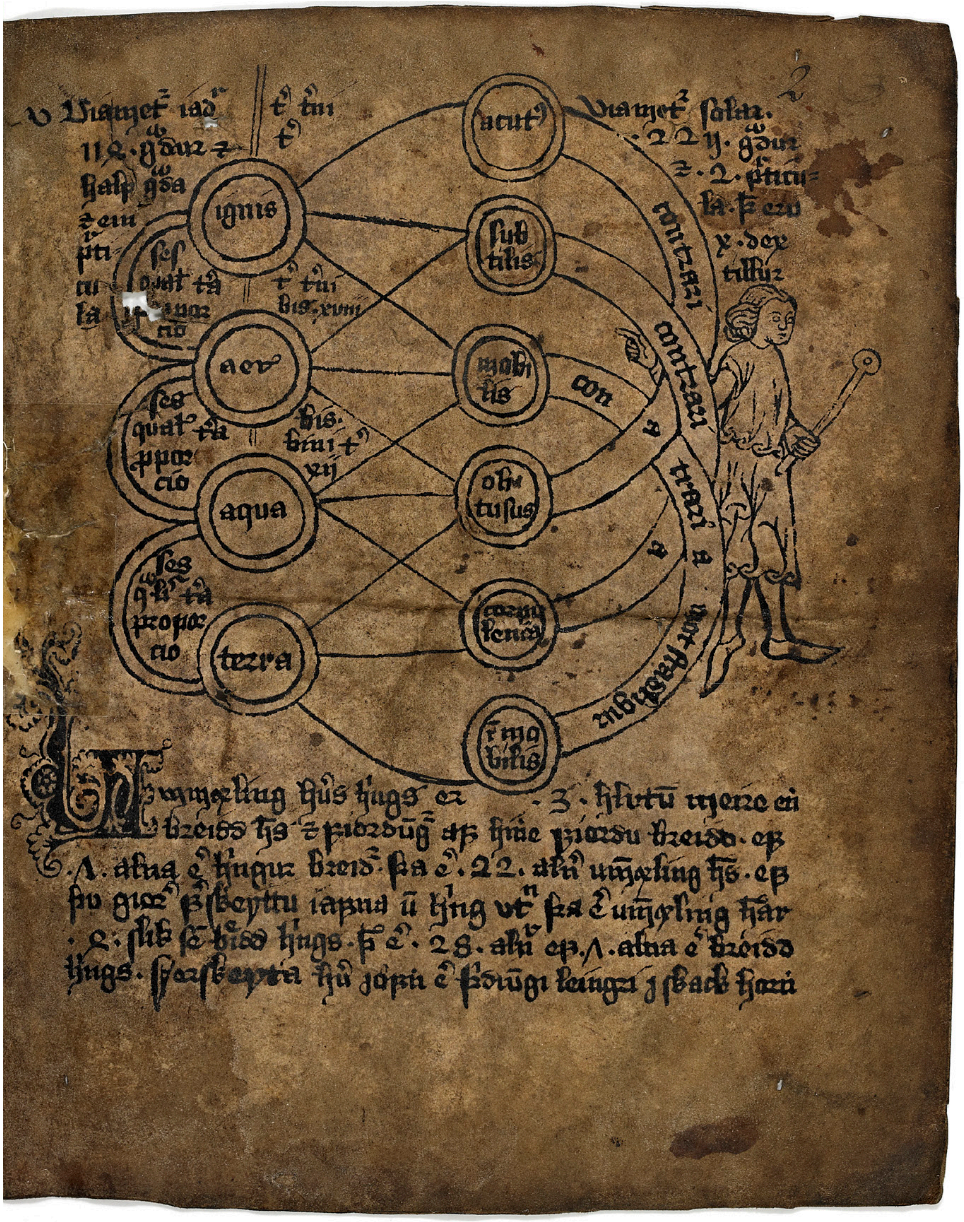

FIGURE 7.3 – A diagram of the four elements in the manuscript AM 736 III 4to, fol. 1r. Photo: Suzanne Reitz.

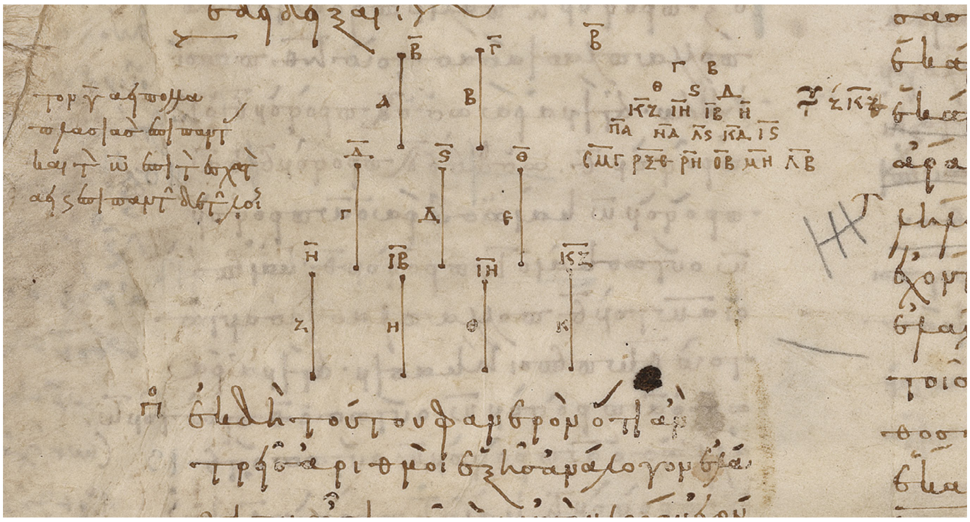

FIGURE 7.4 – A diagram of continued proportions in the manuscript Vat. Gr. 190 (codex P), fol. 115v. © 2021 Biblioteca Apostolica Vaticana.

is also found in the anonymous treatise on cosmology in Bodleian Library Digby 83, fol. 3r.

That particular sequence, 8, 12, 18, 27, appears in a manuscript of Euclid's *Elements*, the Vat. Gr. 190 (codex P), fol. 115v, see Figure 7.4. The Vatican Euclid is a manuscript of a Greek text of Euclid's *Elements* dating from the ninth century.

The diagram in Figure 7.4 is placed at the end of proposition VIII 2 in Euclid's *Elements*. Proposition VIII 2 is the following:

> To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio.[40]

At the top of the vertical line segments in Figure 7.4, marked A, B, Γ, Δ, etc., there are numbers, written in the ancient Greek number notation where alphabetical letters with a bar at the top denote numbers. Thus Ā= 1, and the B and Γ with bars on top of the uppermost vertical line segments denote 2 and 3. At the top of the first vertical line segment in the second row, there is Δ with a bar, denoting 4. Thus, in the top row, the numbers are 2 and 3; in the middle row, 4, 6, 9; and in the bottom row, 8, 12, 18, and 27, the numbers appearing in *Algorismus* and in a number of medieval manuscripts, representing continuous proportions with the ratio 2:3.

40 Euclid, *The Thirteen Books of the Elements*, ed. Thomas L. Heath (New York: Dover, 1956), 2:346.

7.4 The Four Manuscripts of *Algorismus* – Descriptions

The texts of *Algorismus* in the manuscripts AM 544 4to, and GKS 1812 4to are identical in most respects, as is *Algorismus* in AM 685 d 4to, which, however, has added a 306-word long section, placed after the section on subtraction. It describes a method of halving a number, the fourth operation. This section is contained neither in *Carmen* nor in the manuscripts AM 544 4to and GKS 1812 4to, and it is not discussed further in this chapter. The manuscript AM 736 III 4to is fragmentary.

AM 544 4to, part of the manuscript Hauksbók, contains the oldest manuscript of the treatise, estimated to be written in the period 1302–1310, most likely in 1306–1308.[41] The text is divided into chapters with headings. Numbers are written using Hindu-Arabic numerals in the introduction and in the additions to *Carmen* with examples of place-value notation and even and odd numbers, discussed earlier. Numbers are, however, mainly written using Roman numerals, until the last section on the elements, which does not originate in *Carmen de Algorismo*, and where Hindu-Arabic numerals are used.

The part of GKS 1812 4to containing *Algorismus* is estimated to be written in 1300–1400.[42] There are no chapter headings. Numbers are mainly written in words as in *Carmen de Algorismo*, but occasionally in Roman numerals. Hindu-Arabic numerals are only used in the first additions to *Carmen*, as they are in AM 544 4to and in the chapter on the elements.

AM 685 d 4to is dated to 1450–1500.[43] It has no chapter headings. Numbers are written either in words, Roman numerals, or Hindu-Arabic notation, which is the most common. Finnur Jónsson states that the text of *Algorismus* in AM 685 d 4to is the most error-free of the four texts, basing this conclusion on various spelling examples.[44] Furthermore, this text is the most concise of the four texts as it is often contracted, preserving the correct meaning. The text in AM 685 d 4to is also correct where other texts have an error on the origin of one half,[45] called *semiss*, used after halving an odd number, which indicates that one of the transcribers of AM 685 d 4to understood the treatise well.

41 Stefán Karlsson, "Aldur Hauksbókar," *Fróðskaparrit* 13 (1964): 114–121, repr. in Stefán Karlsson, *Stafkrókar*, 303–309.

42 *A Dictionary of Old Norse Prose: Indices* (Copenhagen: The Arnamagnæan Commission, 1989), 26; see also Chapter 3 by Haraldur Bernharðsson in the present volume.

43 *ONP – Indices*, 26.

44 *Hauksbók*, ed. Eiríkur Jónsson and Finnur Jónsson, cxxxi.

45 *Hauksbók*, ed. Eiríkur Jónsson and Finnur Jónsson, 419.

AM 736 III 4to is estimated to originate around 1550.[46] It contains only a fragment of the text of *Algorismus*, a section on root extraction, in addition to the leaf with the diagram of the elements in Figure 7.3.

The adaptations made to *Carmen de Algorismo* to create *Algorismus* suggest that *Algorismus* played a role in introducing the use of Hindu-Arabic numerals in Norse societies. In the oldest manuscript of *Algorismus*, AM 544 4to, Roman numerals are used to explain the text, or plain words are used as in *Carmen*. The use of Roman numerals indicates, on one hand, that the transcriber needed to shorten the text and, on the other, that he was not used to Hindu-Arabic numerals.

Plain-word number notation is dominant in GKS 1812 4to. The youngest whole manuscript, AM 685 d 4to, rarely uses Roman numerals, while words and Hindu-Arabic numerals are used alternately.

7.5 Manuscript Numerical Comparison – Methodology

When reading the four manuscripts of *Algorismus*, it is apparent that they are quite similar; sentence structure and phrasing suggest that they all derive from the same prototype. The same textual insertions and deletions are made in all four manuscripts to *Carmen de Algorismo*, demonstrating that these are not different translations, although the texts in the manuscripts are different in length. Numerical methods were used to compare the manuscripts, comparable to methods that have been used extensively in comparative linguistics[47] and in gene and protein comparison. In the following comparison, difference in spelling is generally not revealed, as the texts of all the manuscripts have been rewritten in modern Icelandic. When the section in AM 685 4to, not extant in the other manuscripts, has been removed, the length of the texts turns out as shown in Table 7.1.

That AM 685 d 4to has fewest words of the complete manuscripts confirms the reader's intuition that the transcriber(s) of AM 685 d 4to sometimes shorten the text.

The four texts were aligned using the computer programme ClustalW[48] and a weighted number of mismatches between the manuscripts was com-

46 *ONP – Indices*, 26.

47 Anthony Fox, *Linguistic Reconstruction: An Introduction to Theory and Method* (Oxford: Oxford University Press, 1995).

48 Thompson, Julie, Desmond G. Higgins, and Toby J. Gibson, "CLUSTAL W: Improving the Sensitivity of Progressive Multiple Sequence Alignment through Sequence Weighting, Position-Specific Gap Penalties and Weight Matrix Choice," *Nucleic Acids Research* 22, no. 22 (1994): 4673–4680.

TABLE 7.1. – Number of words and characters in the four manuscripts of *Algorismus.*

Manuscript	Words #	Characters #
GKS 1812 4to	2986	15174
AM 544 4to	2960	15110
AM 685 d 4to	2902	14772
AM 736 III 4to	630	3323

puted.[49] As an example, a missing word, word insertion, and the use of a different word were counted as one mismatch, while a single character mismatch or different writing style of numbers was counted as a one-quarter mismatch. The shortest distance between two manuscripts is between AM 544 4to and GKS 1812 4to, 123¼ mismatches or 4.1 percent, while the greatest distance is between AM 685 4to and GKS 1812 4to, 264½ mismatches or 8.9 percent.

Figure 7.5 shows the relation between the different copies of *Algorismus*. A matrix was made according to the distances between the parts of the four manuscripts that they have all in common, that is the part also found in AM 736 III 4to, from which a phylogenetic tree was constructed with distances similar to the distances in the distance matrix. The diagram was made by the programme ATV.[50]

The phylogeny may be interpreted in the following way. The copy of the *Algorismus* in AM 544 4to is, on the whole, closest to the original, followed by the copy in GKS 1812 4to. The copies in the manuscripts AM 736 III 4to and AM 685 d 4to are partly drawn from the same stem but are further from the origin, especially AM 736 III 4to.

The difference in the number of words between the two copies, GKS 1812 4to and AM 544 4to, is twenty-six words – GKS 1812 4to being the longer of the two. Of these words, eighteen can be ascribed to the differently expressed imperative form of the verbs, such as *skalt þú* – *skaltu*. The contracted form is more common in AM 544 4to while the uncontracted form is the norm in GKS 1812 4to.

49 For further details, see Kristín Bjarnadóttir and Bjarni V. Halldórsson, "The Norse Treatise Algorismus," in *Actes du 10ème colloque maghrébin sur l'histoire des mathématiques Arabes (Tunis, 29–30–31 mai 2010)*, ed. Mahdi Abdeljaouad et al. (2010), 67–77.

50 Christian M. Zmasek and Sean R. Eddy, "ATV: Display and Manipulation of Annotated Phylogenetic Trees," *Bioinformatics* 17, no. 4 (2001): 383–384.

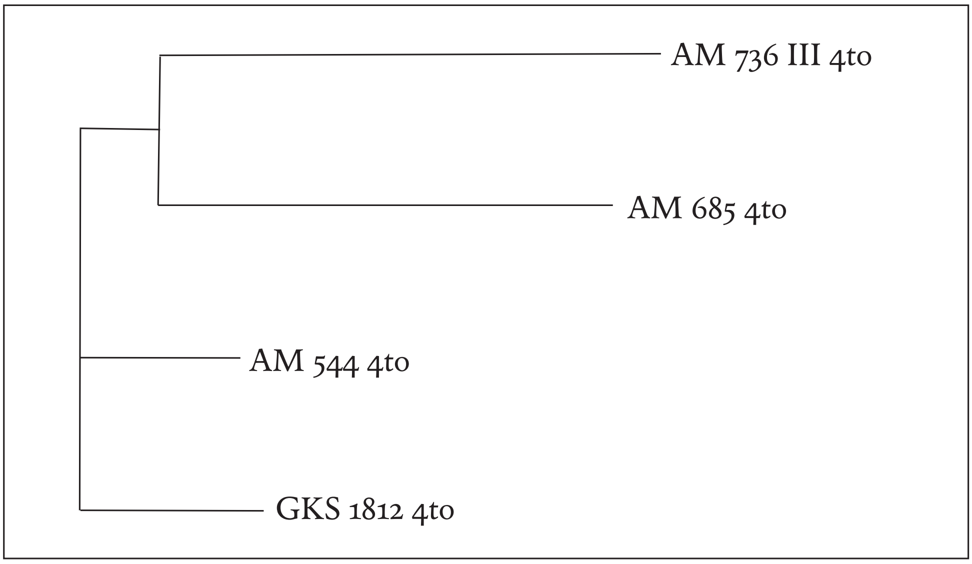

FIGURE 7.5 – A phylogeny of the copies of *Algorismus* in the manuscripts AM 736 III 4to, AM 685 d 4to, AM 544 4to, and GKS 1812 4to, made by the programme ATV.

It is not unreasonable to conclude that the two versions of *Algorismus* are copies of the same original, possibly the first or second copies of the original version. The two versions of *Algorismus* are similar in length. Both contain several of the same errors, for example in the section about doubling:

> … en ef semiss stendur yfir uppi í ysta stað þá legg við einn því að þar var áður jöfn tala er í helminga var skipt.[51]
>
> (… but if a semiss [symbol for one half] is placed above in the farthest place then add one as there was earlier an even number divided into halves.)

Here, *even* [number] should be replaced by *uneven/odd* [number]. This error is not found in the copy contained in the fifteenth-century manuscript AM 685 d 4to.

7.6 Conclusion

We have explored the thirteenth-century *Algorismus*, written in Old Norse, its content of arithmetic studies and cosmology, and compared the surviving manuscripts that contain the treatise. *Algorismus* was written in

51 GKS 1812 4to, 14r.

the transition period when Hindu-Arabic number notation and associated arithmetic methods were being introduced in Middle and Northern Europe. The four different manuscripts of *Algorismus*, written during the time span from the early fourteenth century until the mid-sixteenth century, reveal that the new style of number notation took hold gradually. We may wonder how large a role *Algorismus* played in that development.

The main topic of the treatise is a translation of the Latin hexameter *Carmen de Algorismo* on arithmetic by the thirteenth-century French scholar Alexander de Villa Dei. This translation is slightly extended by clarifications from the Icelandic writer on the original text. Furthermore, its content about drawing the cubic root seems to have enkindled association with ancient cosmological ideas, circulating in the medieval world, about the cubic numbers 8 and 27, associated with the elements of earth and fire, and further producing the numbers 12 and 18, representing water and air. This section about the elements is considered the only known incidence of a reference in Old Norse to the *Timaeus* of Plato, most likely derived from the Latin translation by Calcidius.[52] The cosmological ideas seem also to be related to a theory on proportions, presented in the ancient textbook *Elements* by Euclid from 300 BC, and elaborated upon in a ninth-century manuscript Vat. Gr. 190 (codex P) fol. by the number sequence 8, 12, 18, 27.

Another aspect to notice is the relation between *Carmen de Algorismo* from 1202, *Algorismus vulgaris* by Johannes de Sacrobosco from around 1225, and the Old Norse *Algorismus*, from ca. 1270. Sacrobosco cites three verses from *Carmen* as mentioned earlier. The Icelandic author of *Algorismus* seems to have known Sacrobosco's work when he added a phrase on two ways of extracting roots to his translation of *Carmen*. Finally, the shape of the numerals in the manuscripts of Algorismus is ascribed to Sacrobosco.

What motivated the translation of *Carmen de Algorismo* from Latin into Old Norse? Certainly, people had to count their belongings and assets, e.g. for taxes, but they could have done that with the Roman numerals they already knew. The Old Norse-speaking population in Iceland and Norway was never large compared to other language groups in Christendom. Producing texts in the vernacular was an important factor in creating a common culture of this small group of people.

The additions to the original Latin *Carmen de Algorismo* in the Old Norse *Algorismus* bear witness to a desire for learning and comprehending

52 Olaf Pedersen, "Matematisk litteratur," *Kulturhistorisk leksikon for nordisk middelalder,* 11:498.

the source text, as demonstrated by the Icelandic scribes' insertions for clarification. Comparison of the four copies of *Algorismus* from different periods reveals that there was a continual need to work on understanding the text and that the scribes gradually began to use the convenient Hindu-Arabic number notation.

This took time. According to the phylogeny and other considerations, the manuscript AM 544 4to was not the original of *Algorismus*, which suggests that *Algorismus* may have been written in the second half of the thirteenth century, as proposed by Finnur Jónsson, or about 200 years before AM 685 d 4to, and possibly up to 300 years before AM 736 III 4to. *Algorismus* therefore played an important role in Icelandic culture until the era of printing, when printed books began to spread much more rapidly between countries than manuscripts.

Algorismus has appeared throughout Icelandic history whenever mathematical education underwent a revival, serving as a monument of the moment in the proud past when Icelanders had kept up with the latest global knowledge and translated it into their own language. Even the most distinguished Icelandic scholars continued to refer to *Algorismus* up until the nineteenth century.[53]

53 "Að þeir sè indverskir, sjá Hauks Erlendssonar Algorithmus." (On the fact that they are Indian numbers, see the Algorithmus of Haukr Erlendsson.) Björn Gunnlaugsson, *Tölvísi* (Reykjavík: Hið íslenzka bókmenntafélag, 1865), 4.